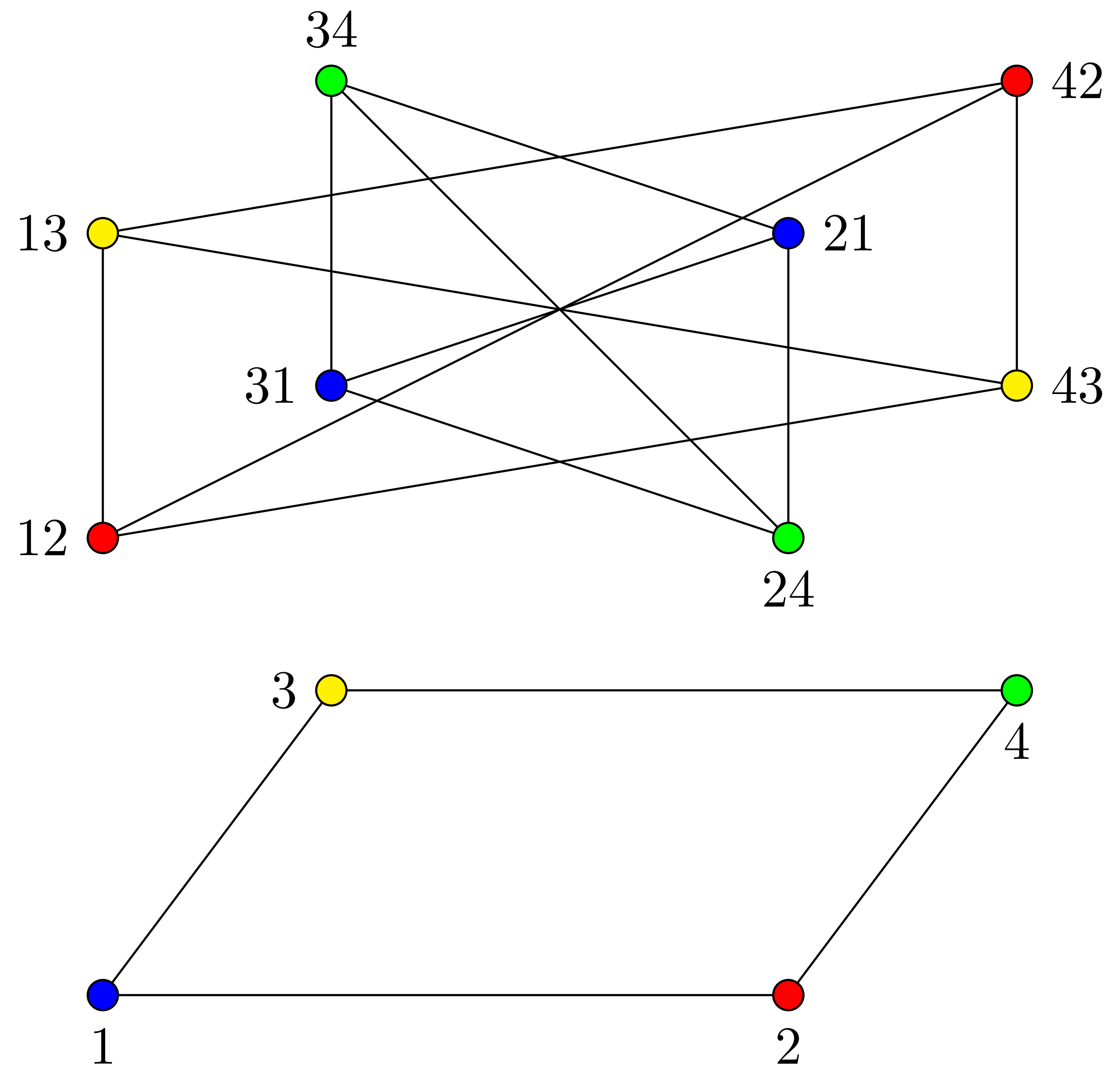
Graphs and Homotopy Theory



NYC Combinatorics Day

4/17/26

Emilio Minichiello

NEW YORK CITY COLLEGE OF TECHNOLOGY
CITY TECH

A little about me...

Emilio Minichiello

Previously:

Got my PhD at the CUNY Graduate Center

- Studied Topology and Category Theory

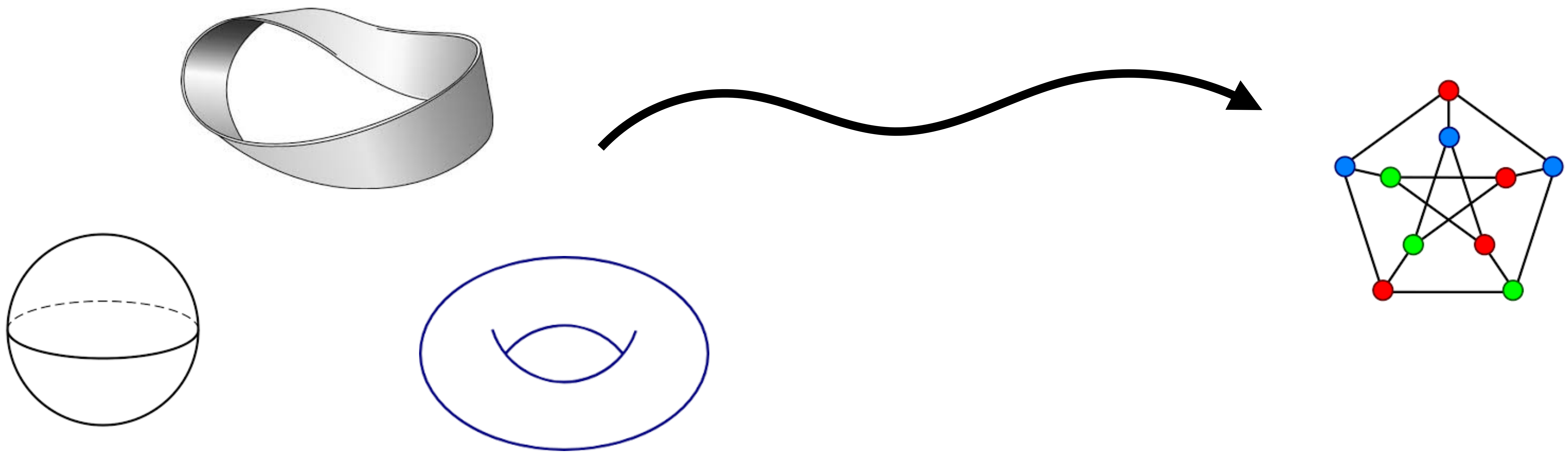
- Advisor: Mahmoud Zeinalian

Now:

Assistant Professor at CUNY CityTech

Question:

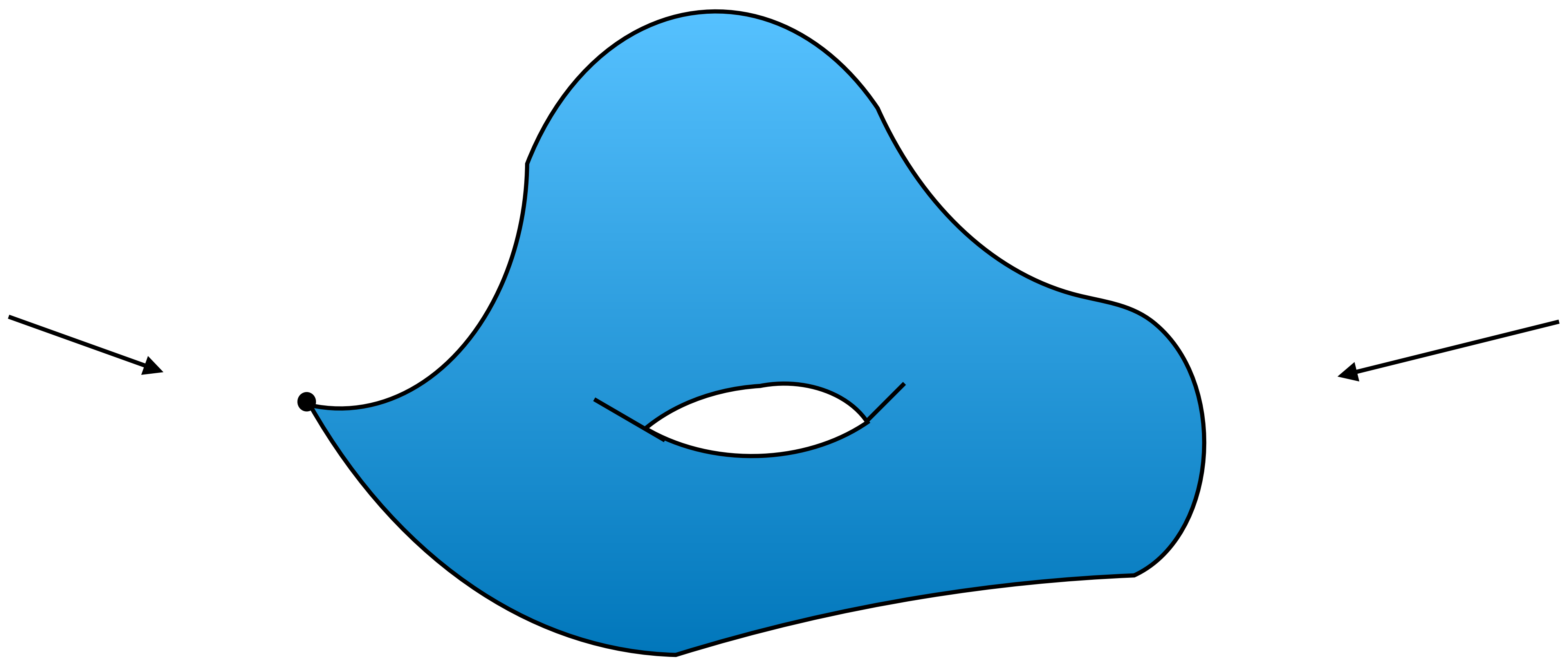
What can topology tell us about graph theory?



Given a space X, we can associate to it **topological invariants**, namely its **homotopy groups** π_n and **homology groups** H_n.

These "count" the n-dimensional "holes" of the space.

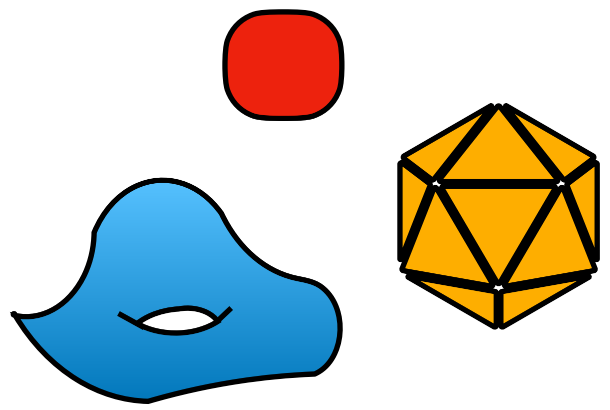
$$\begin{array}{c} \vdots \\ \pi_3(X) \\ \pi_2(X) \\ \pi_1(X) \\ \pi_0(X) \end{array}$$

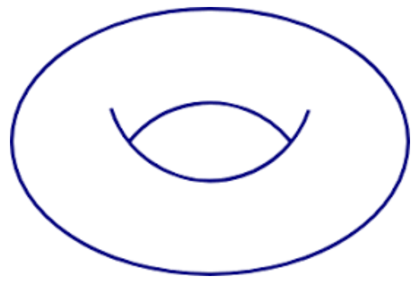


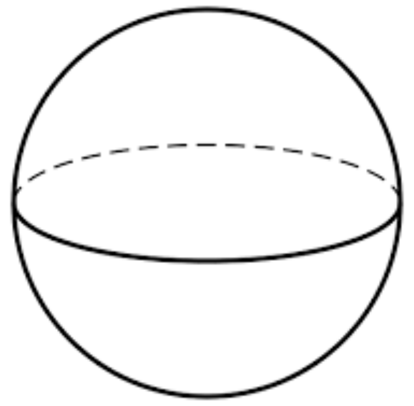
$$\begin{array}{c} \vdots \\ H_3(X) \\ H_2(X) \\ H_1(X) \\ H_0(X) \end{array}$$

Let $\mathrm{Conn}(X)$ of a space X denote the lowest k for which $\pi_k(X)$ (and consequently $H_k(X)$) is nonzero.

Connectivity Conventions







Every Space	Nonempty Space	Connected Space	Simply Connected Space
-2 Connected	-1 Connected	0 Connected	1 Connected

We call a continuous function $f : X \to Y$ between spaces a **weak equivalence** if it induces an isomorphism $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ on all of the homotopy groups (and consequently the homology groups).

The idea here is that we only care about spaces up to continuous deformation.

Image from TomRocksMath

Idea of **topological combinatorics**:

Given a combinatorial object G (e.g. a graph), construct a space X out of it

$\downarrow$

Study the topology of the space X using invariants

$\downarrow$

Learn something about the combinatorial object G

I'm going to talk about the main theorem from my recent paper

"Thomason-Type Model Structures on Simplicial Complexes and Graphs"

Published in Applied Categorical Structures Volume 34, article number 12, (2026) I'm going to talk about the main theorem from my recent paper

"Thomason-Type Model Structures on Simplicial Complexes and Graphs"

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But first, I want to talk about some motivation.

My paper sprang out of an attempt to understand Lovász's Theorem (1978) through the lens of modern homotopy theory.

Theorem:[Lovász, 1978]

Given a loop graph G,

$$\chi(G) \geq \mathrm{Conn}\left(\mathrm{C}\ell\left(\mathrm{Lov}(G)\right)\right) + 3$$

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Topological Connectivity

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Chromatic Number

Clique Complex

Lovász Graph

Theorem:[Lovász, 1978]

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Topological Connectivity

Chromatic Number

Clique Complex

Lovász Graph

Let's explain all of this means!

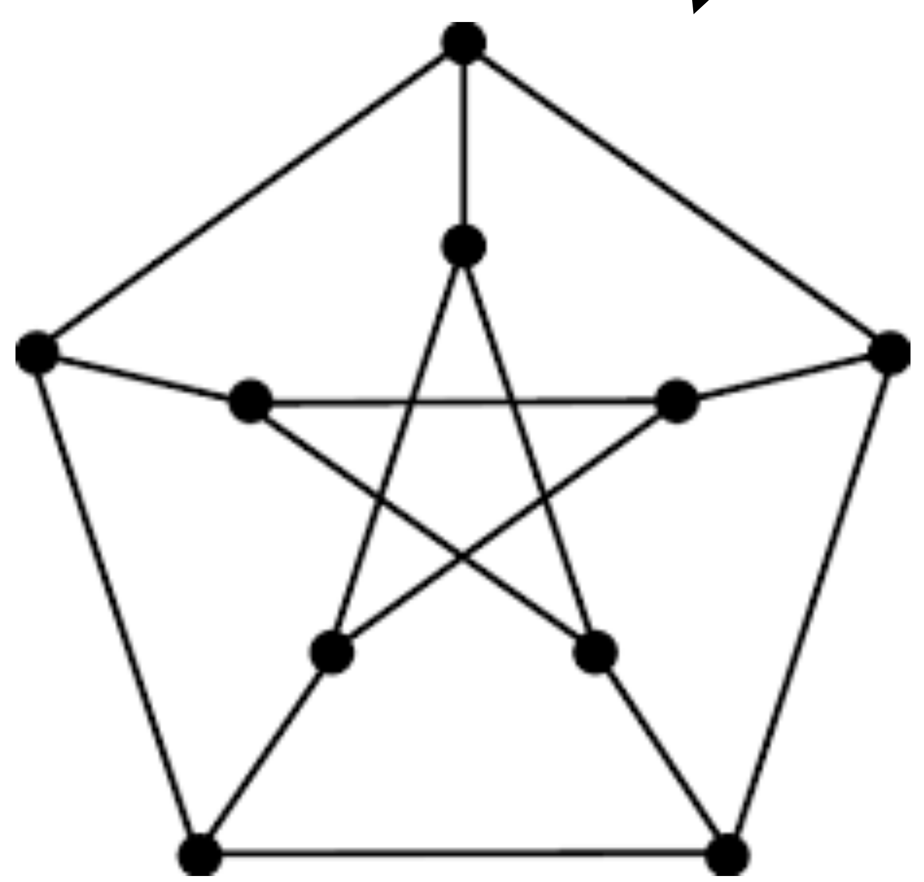
Note: Lovasz originally used a different construction, called the neighborhood complex

Dochtermann (2008) showed that this is equivalent to the previous slide.

Chromatic Number:

How many colors do I need

to color every vertex such that:



No vertices of the same color are adjacent

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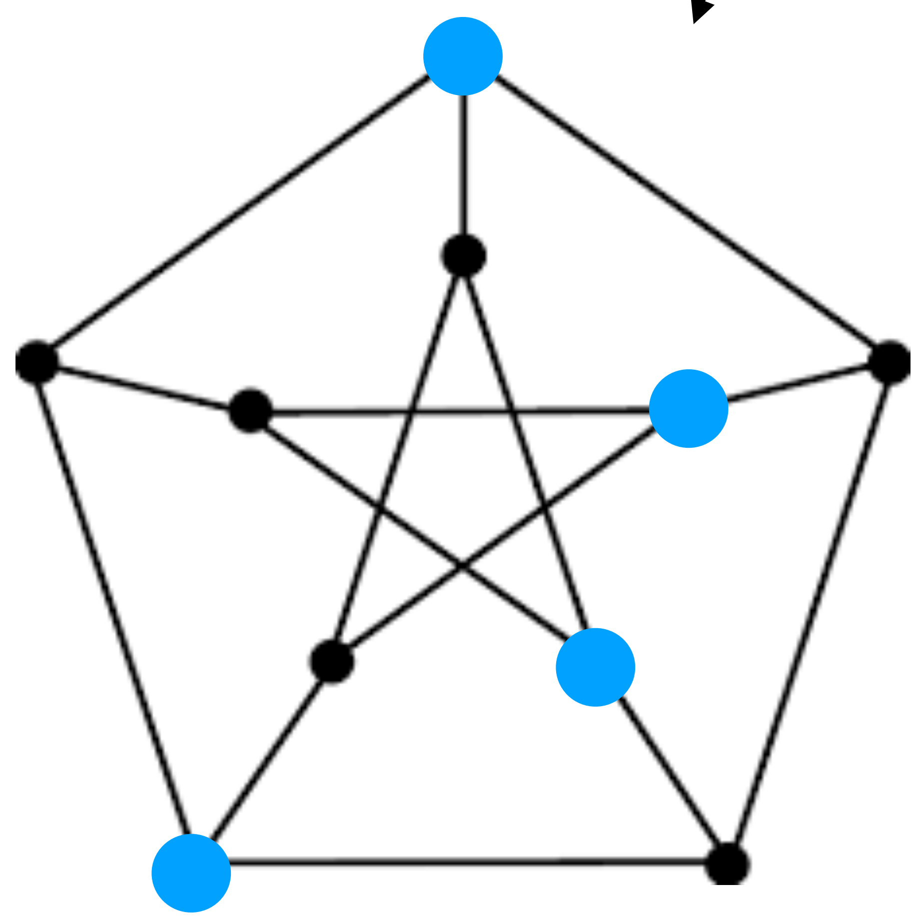

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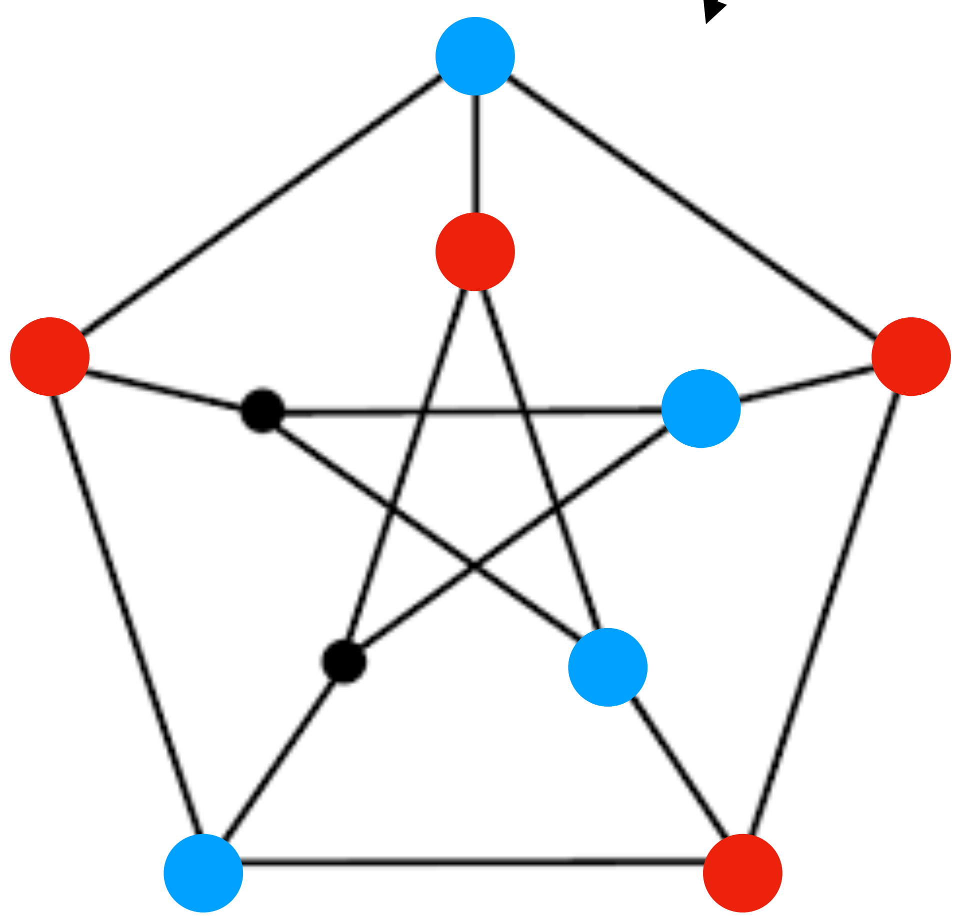

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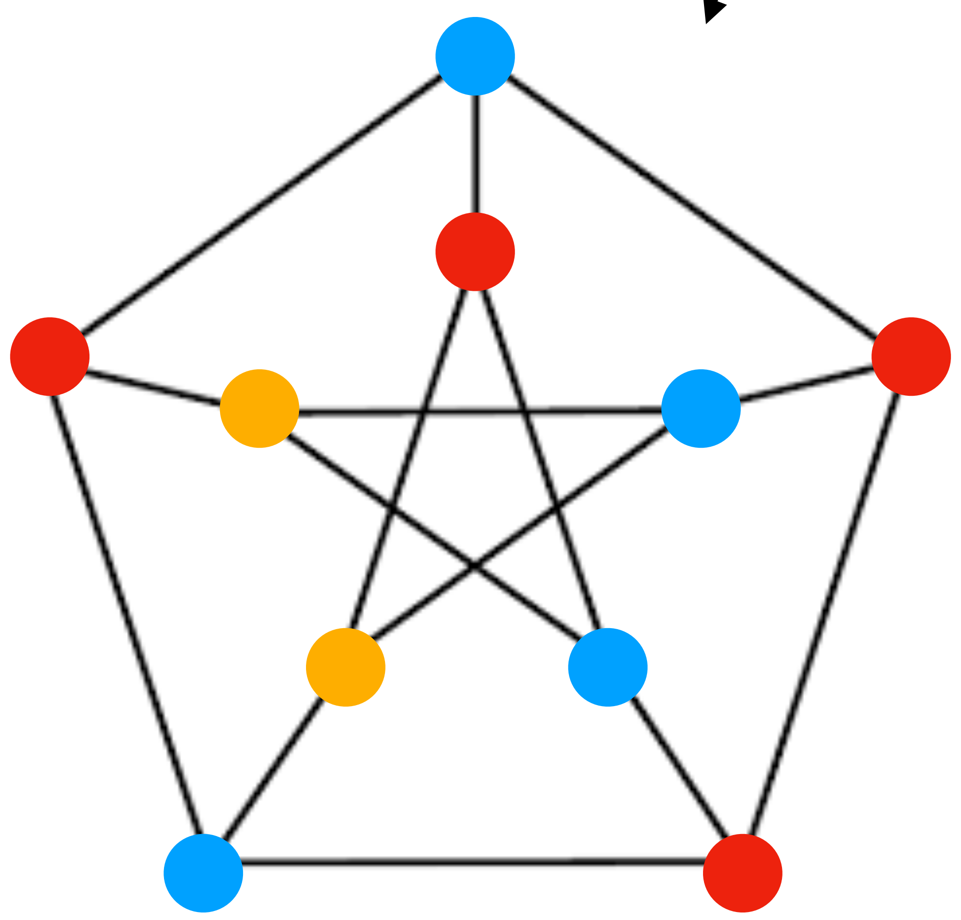

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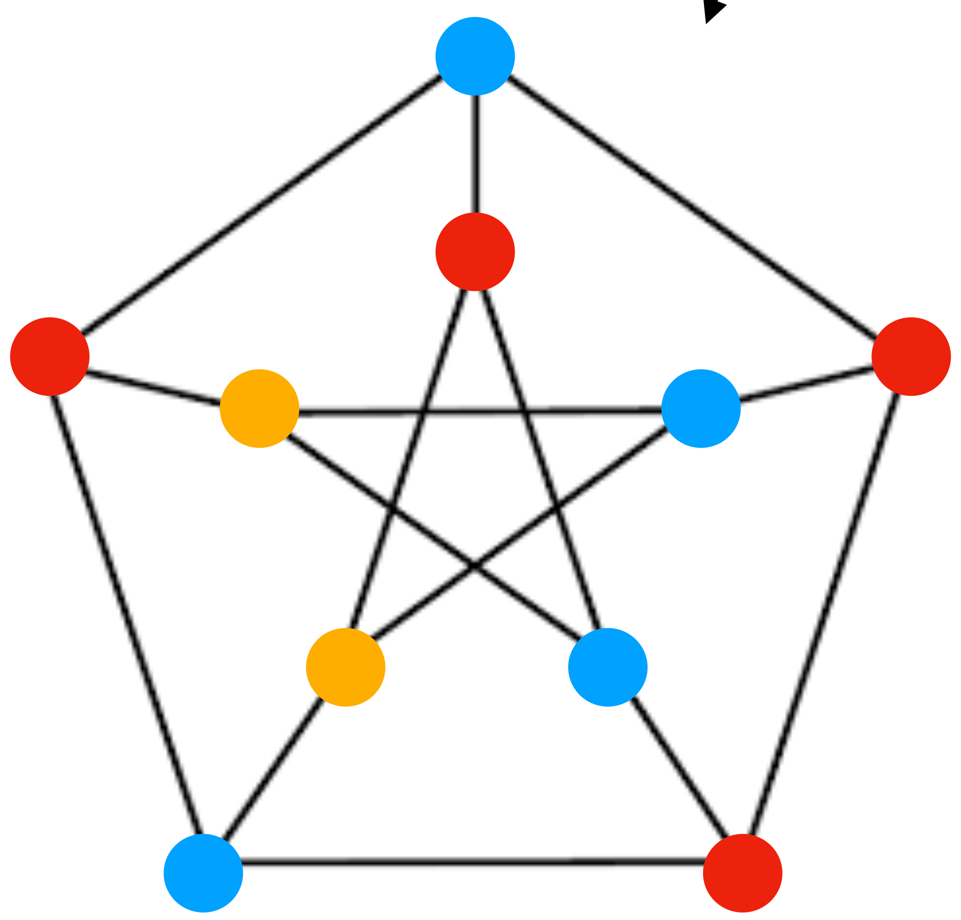


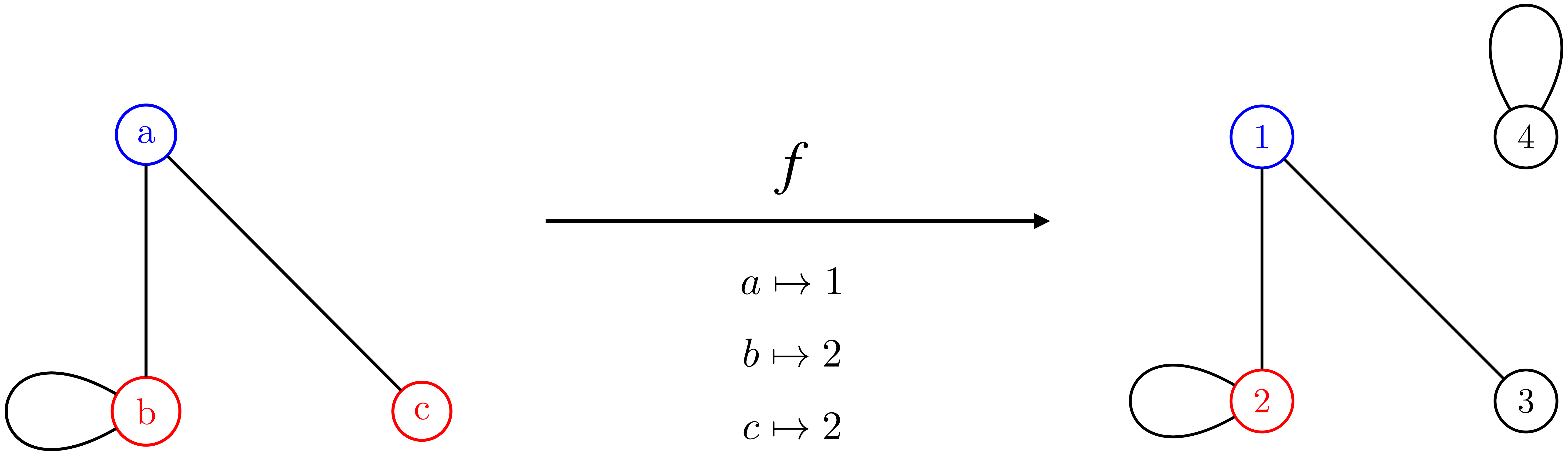
No vertices of the same color are adjacent

We let $\chi(G)$ denote the smallest number n such that G is n-colorable.
We call this the **chromatic number** of G.

Def: An (undirected) **loop graph** G consists of a set $V(G)$ along with a symmetric relation $E(G) \subseteq V(G) \times V(G)$.

We call the elements of $V(G)$ **vertices**. Given vertices x, y, we write $x \sim y$ if $(x, y) \in E(G)$. We say x and y are **adjacent**.

A loop graph homomorphism $f : G \to H$ consists of a function $V(f) : V(G) \to V(H)$ such that if $x \sim y$, then $f(x) \sim f(y)$.



Def: A **category** $\mathcal{C}$ consists of a collection of **objects** and **morphisms**.

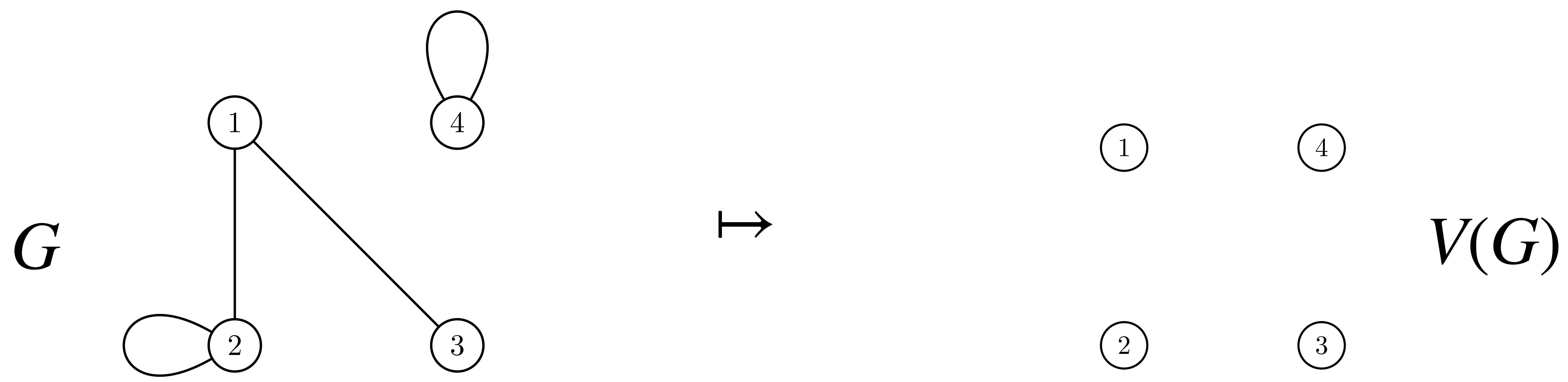
Every object must have an identity morphism, and there must be a notion of composition of morphisms.

Ex:

- The category **Set** whose objects are sets and morphisms are functions,
- The category **Vect** of vector spaces and linear maps,
- The category **Top** of topological spaces and continuous functions,
- The category $\mathbf{Gr}_\ell$ of loop graphs and loop graph homomorphisms.

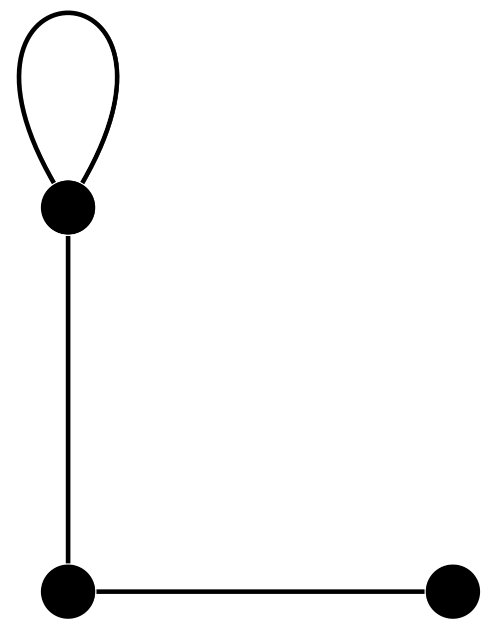
Def: A **functor** $F : \mathcal{C} \to \mathcal{D}$ between categories assigns objects to objects and morphisms to morphisms, in a way that preserves identity morphisms and composition.

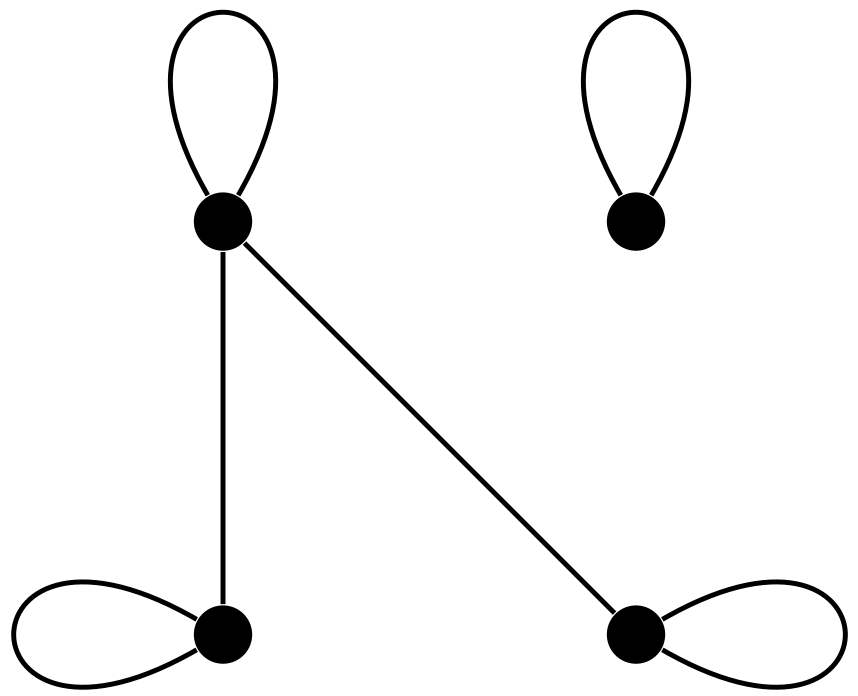
Ex: Simple example $V : \mathbf{Gr}_\ell \to \mathbf{Set}$. Takes a loop graph G to its set of vertices $V(G)$ and a loop graph homomorphism $f : G \to H$ to its underlying set function $V(f) : V(G) \to V(H)$.



Def: We say a loop graph G is **reflexive** if every vertex has a loop. We let $\mathbf{Gr}_r$ denote the subcategory of reflexive loop graphs.

Not reflexive





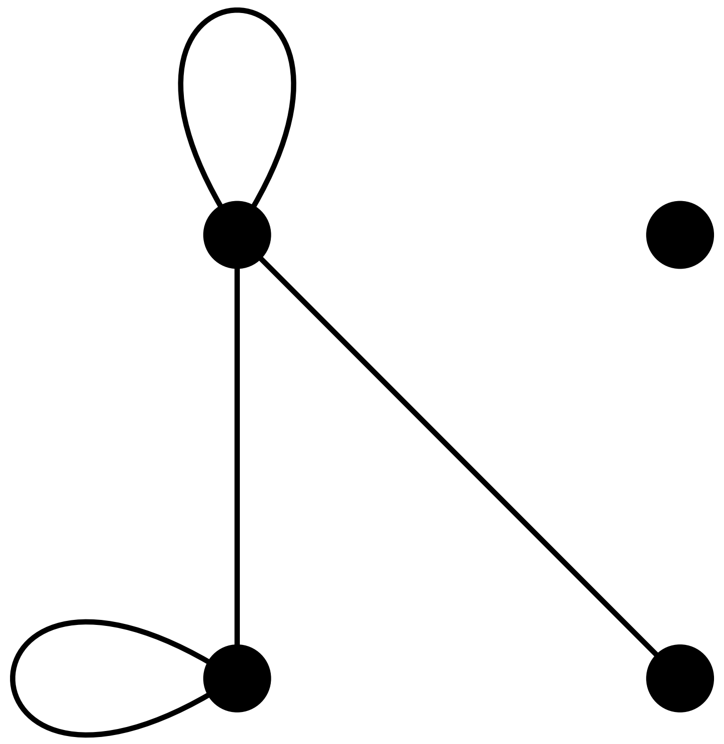
Reflexive

Get a functor $\mathbf{Gr}_r \hookrightarrow \mathbf{Gr}_\ell$ just given by inclusion of reflexive graphs into all loop graphs.

There is also a functor going in the opposite direction

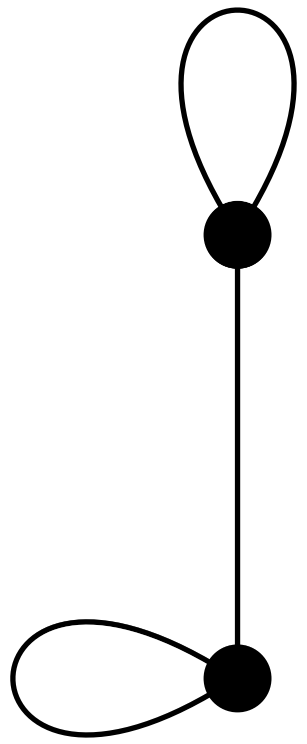
$$\mathbf{Gr}_\ell \xrightarrow{(-)^\circ} \mathbf{Gr}_r$$

If G is a loop graph, then G° is its **maximal reflexive subgraph**.



G

$\mapsto$



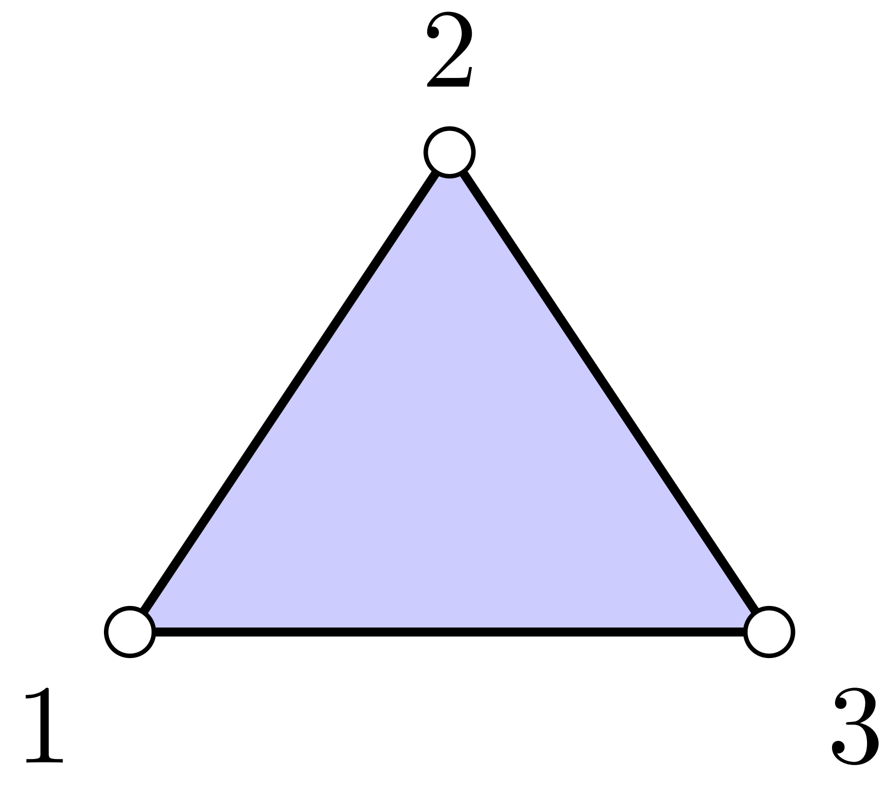
G°

Okay, so now we need simplicial complexes...

Def: Given a set $V(\Delta)$, a **simplicial complex** on $V(\Delta)$ consists of a collection Δ of subsets of $V(\Delta)$ such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.

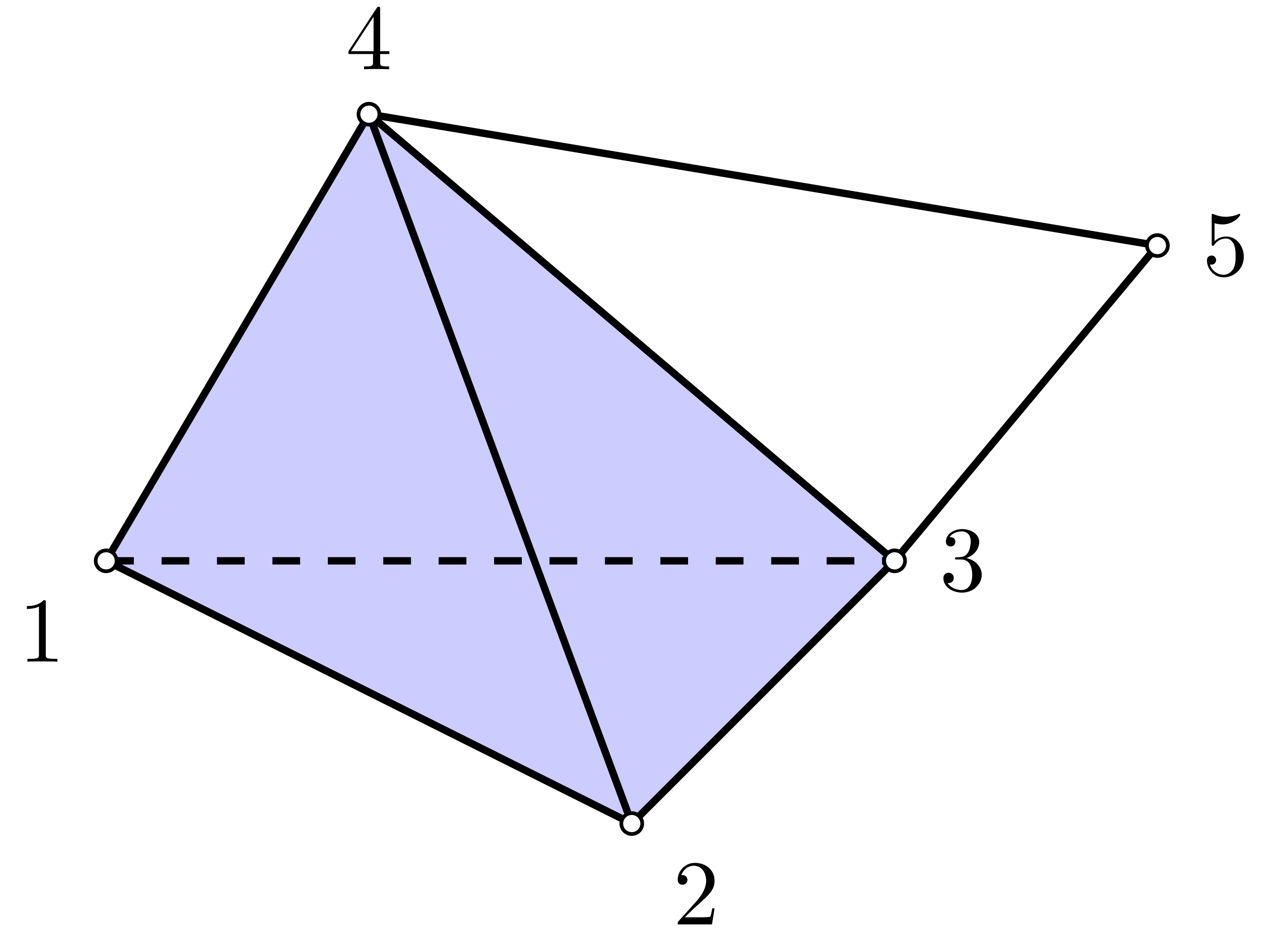
$$V(\Delta) = \{1, 2, 3\}$$

$$\Delta = \{1, 2, 3, 12, 13, 23, 123\}$$

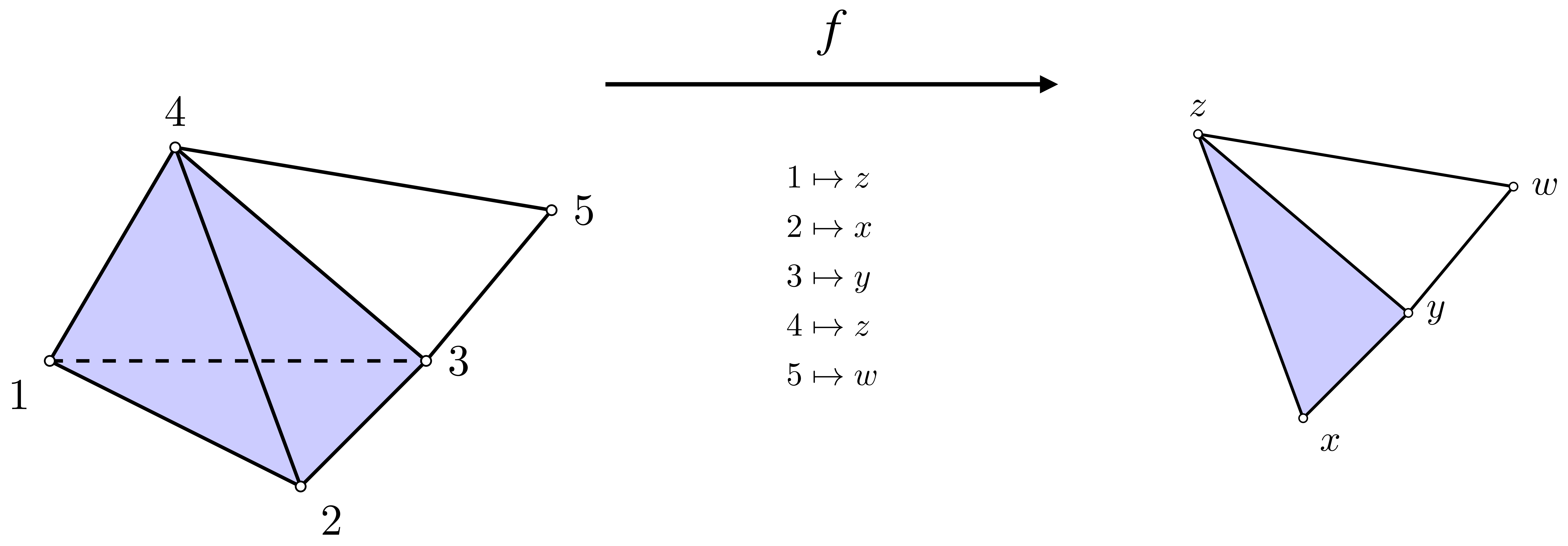


$$V(\Delta) = \{1, 2, 3, 4, 5\}$$

$$\Delta = \{1, 2, 3, 4, 5, 12, 13, 14, 23, 24, 35, 45, 124, 123, 134, 234, 1234\}$$



Def: A morphism of simplicial complexes $f : \Delta \to \Delta'$ consists of a function $V(f) : V(\Delta) \to V(\Delta')$ such that if $\sigma \in \Delta$, then $f(\sigma) \in \Delta'$.



Let **Cpx** denote the category of simplicial complexes.

Get several functors

$$
\begin{tikzcd}
\mathbf{Gr}_r \arrow[r, bend left=50, "\iota"] \arrow[r, bend right=50, "\mathrm{C}\ell"'] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"] & \mathbf{Cpx} \arrow[l, "(-)_{\leq 1}" description]
\end{tikzcd}
$$

Let **Cpx** denote the category of simplicial complexes.

Get several functors

Inclusion

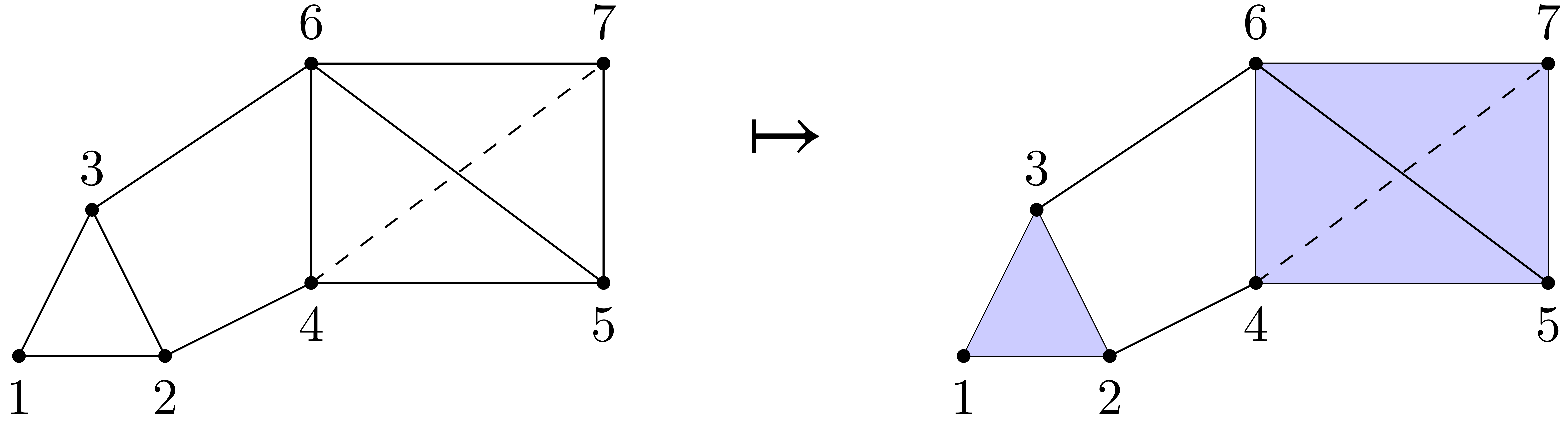
$$\begin{tikzcd} \mathbf{Gr}_r \arrow[r, bend left, "\iota"] \arrow[r, bend right, "\mathrm{C}\ell"'] & \mathbf{Cpx} \arrow[l, "(-)_{\leq 1}"' description] \end{tikzcd}$$

$\perp$ $\perp$

Clique Complex

Take just the edges and vertices

Given a graph G, its **Clique complex** $\mathrm{C}\ell(G)$ is the simplicial complex, where we fill in every complete subgraph (clique) with a simplex.

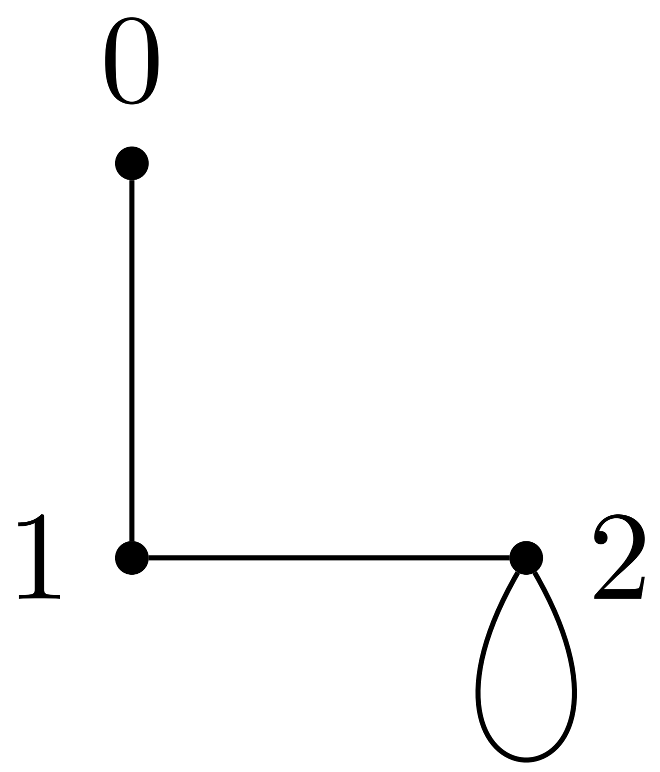


Summarizing:

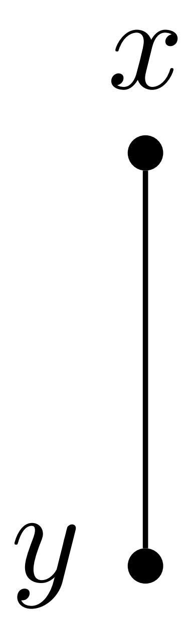
$$
\begin{tikzcd}
\mathbf{Gr}_\ell \arrow[r, bend right, "(-)^\circ"'] \arrow[r, phantom, "\perp"] & \mathbf{Gr}_r \arrow[l, bend right, "i_\ell"'] \arrow[r, bend right, "\mathrm{C}\ell"'] \arrow[r, phantom, "\perp"] & \mathbf{Cpx} \arrow[l, bend right, "(-)_{\leq 1}"']
\end{tikzcd}
$$

Need one last construction:

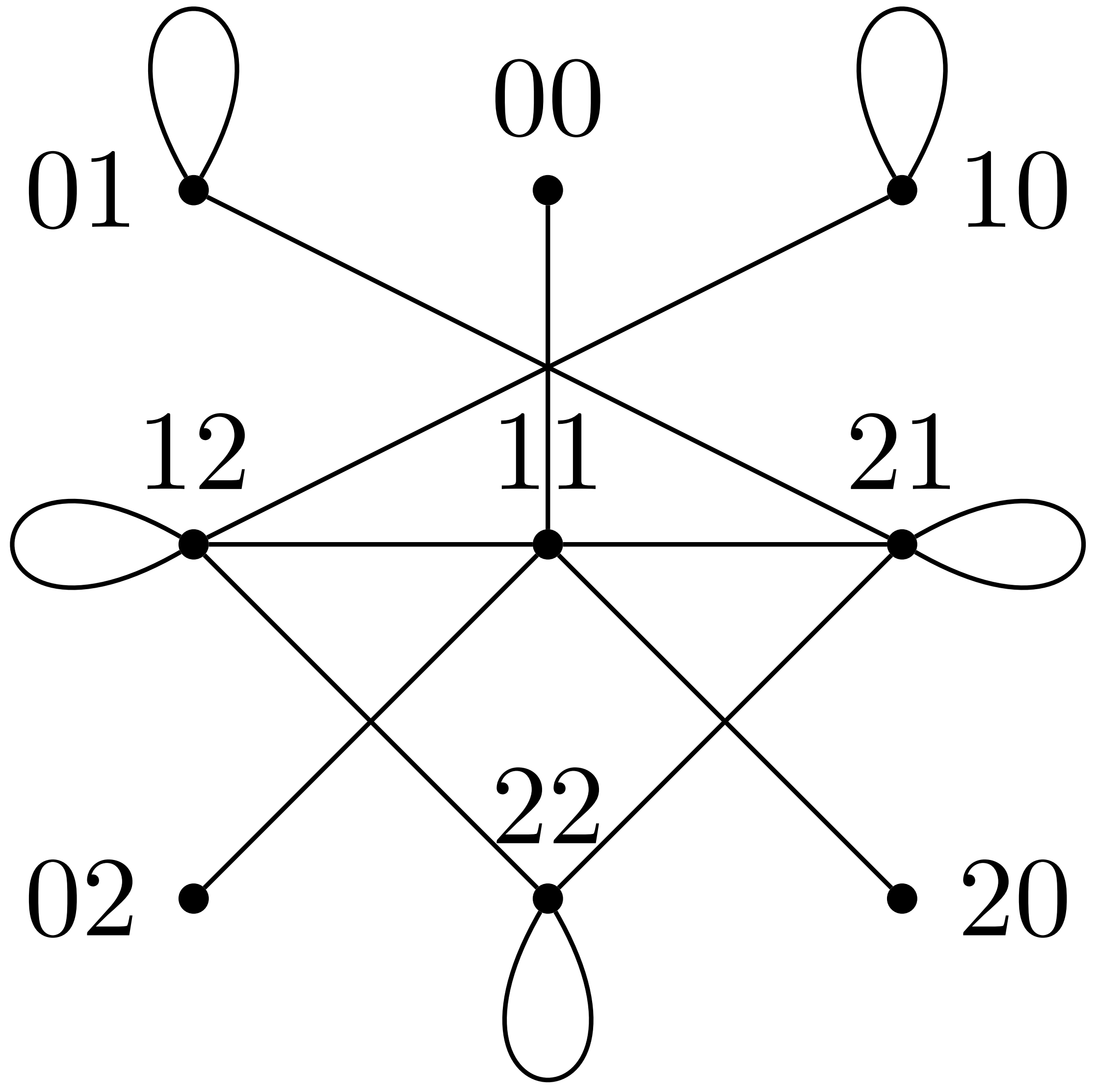
Def: Given loop graphs G and H, let H^G denote their **exponential graph**. This is the graph whose vertices are functions $f : V(G) \to V(H)$, and $f \sim f'$ if for every edge $x \sim y$ in G, $f(x) \sim f(y)$ in H.



H



G



H^G

Its not super important to understand this definition. The graph H^G is completely determined by the following property:

Given a loop graph K, there is a bijection

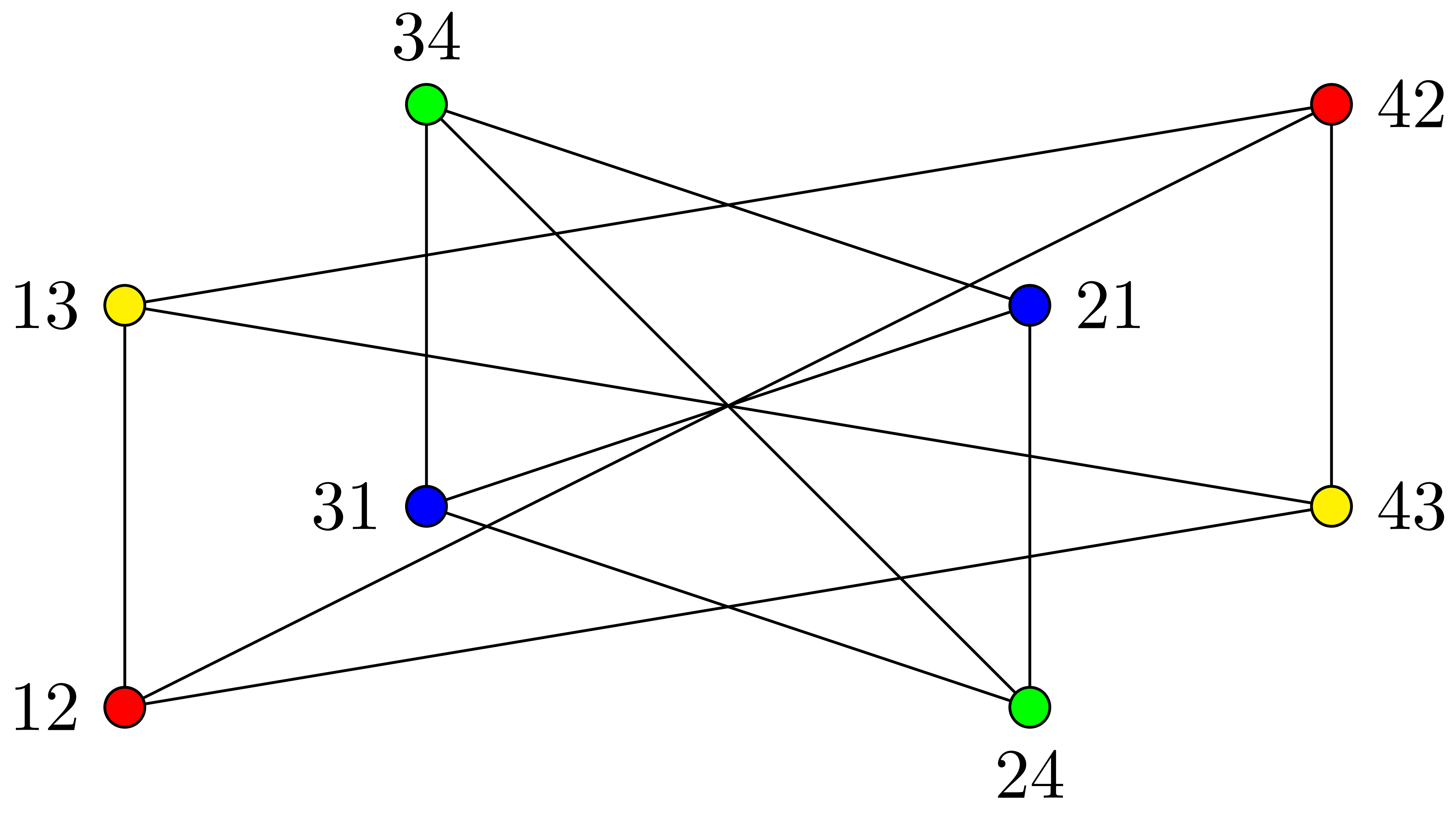
$$\{K \to H^G\} \qquad \leftrightarrow \qquad \{K \times G \to H\}$$

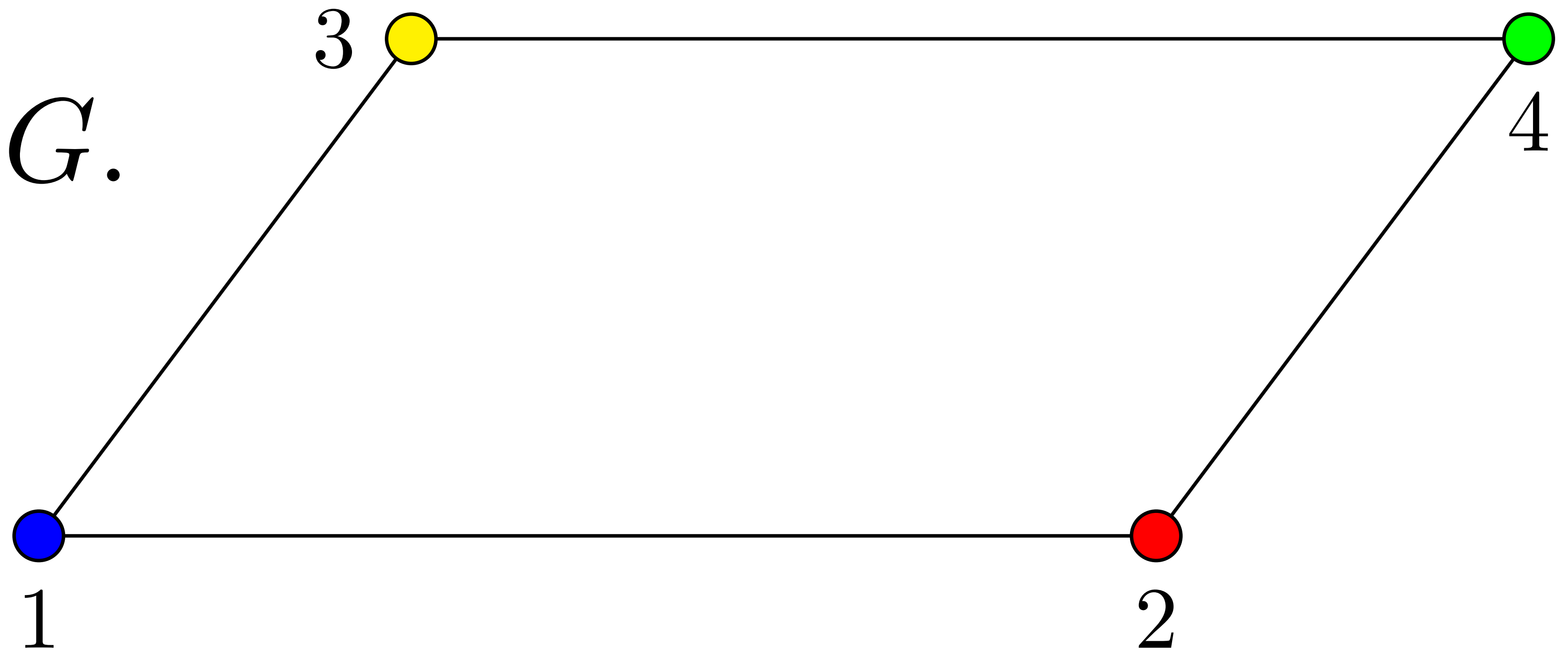
This is called the **internal hom** in the category $\mathbf{Gr}_\ell$.

Can now describe the Lovász graph

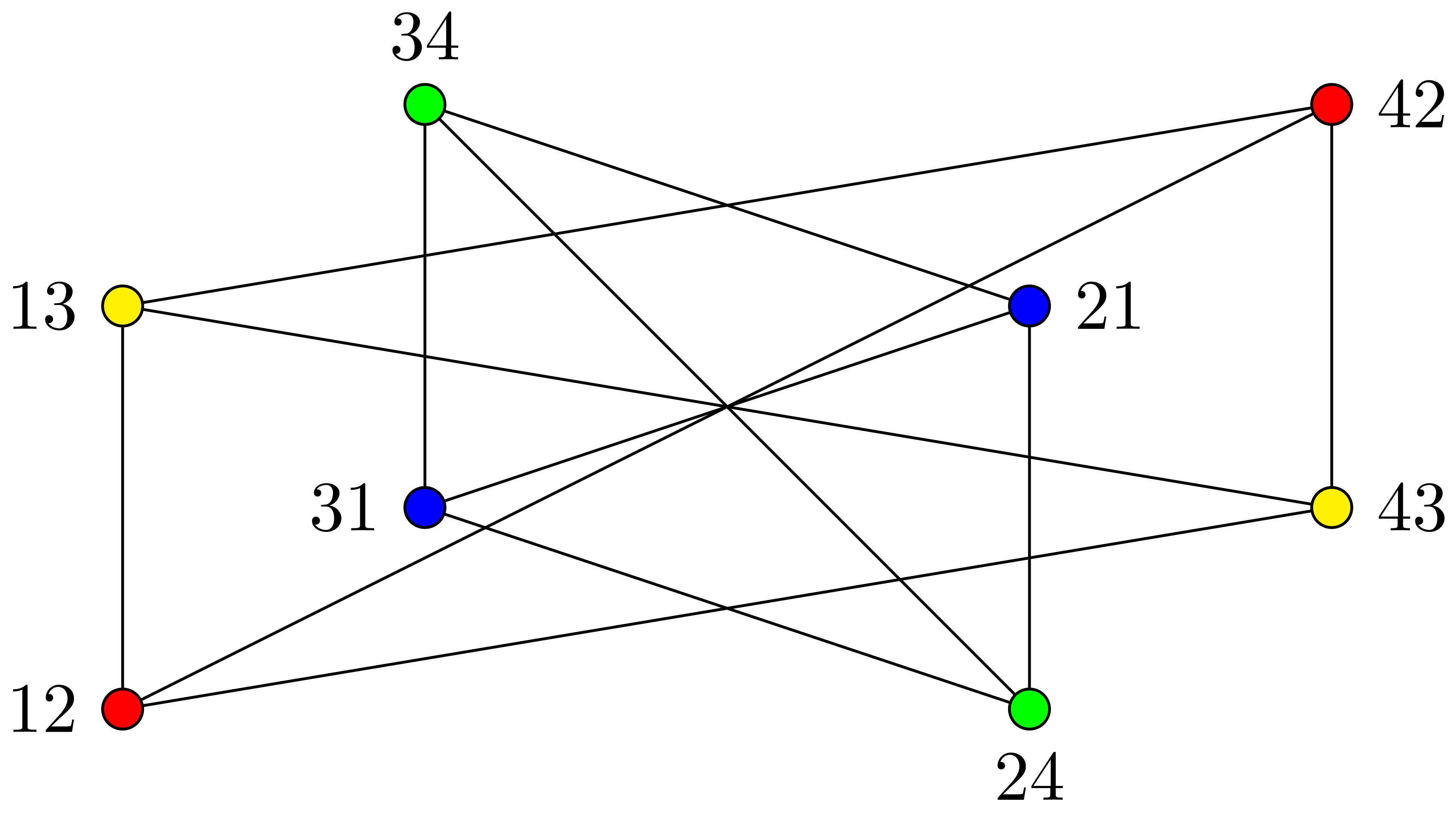
$$\mathrm{Lov}(G) = \left(G^{K_2}\right)^{\circ}$$

When $G = C_4$, $\mathrm{Lov}(G)$ is:





Vertices of $\mathrm{Lov}(G)$ are the directed edges of G.
Two vertices in $\mathrm{Lov}(G)$ are adjacent
if opposite ends are adjacent in G.



Since $\mathrm{Lov}(C_4)$ is disconnected, so is $\mathrm{C}\ell(\mathrm{Lov}(C_4))$, hence $\mathrm{Conn}(\mathrm{C}\ell(\mathrm{Lov}(C_4))) = -1$. This gives a lower bound by Lovász' theorem,

$$\chi(C_4) \geq \mathrm{Conn}(\mathrm{C}\ell(\mathrm{Lov}(C_4))) + 3 = -1 + 3 = 2,$$

which is tight in this case.

So I was learning about the Lovasz Theorem, and I wondered if the following was true:

$$\mathrm{C}\ell((H^G)^\circ) \simeq \underline{\mathbf{Top}}(\mathrm{C}\ell(G), \mathrm{C}\ell(H))$$

Weak equivalence

Space of continuous functions

In the naive sense, I knew it was false.

But I wondered if there was a **model structure** on loop graphs such that

$$\mathbb{R}\mathrm{Hom}(G, H) \simeq \mathrm{C}\ell(H^G)$$

A model structure on a category $\mathcal{C}$ consists of collections of morphisms called weak equivalences, fibrations and cofibrations that interact in a special way.

They are a way of obtaining a homotopy theory in a general category.

Think of them like a toolkit for studying invariants of objects.

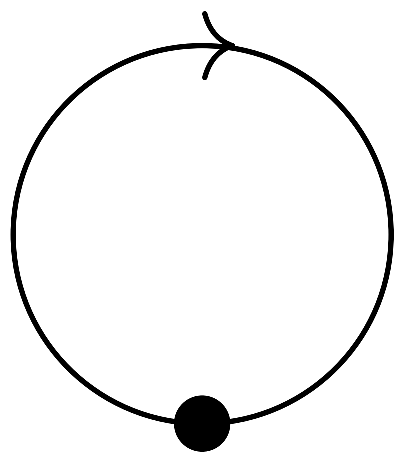
Given a category $\mathcal{C}$ with a model structure, called a **model category**, for any pair of objects $c, d \in \mathcal{C}$ there is a space one can cook up called the **derived mapping space**

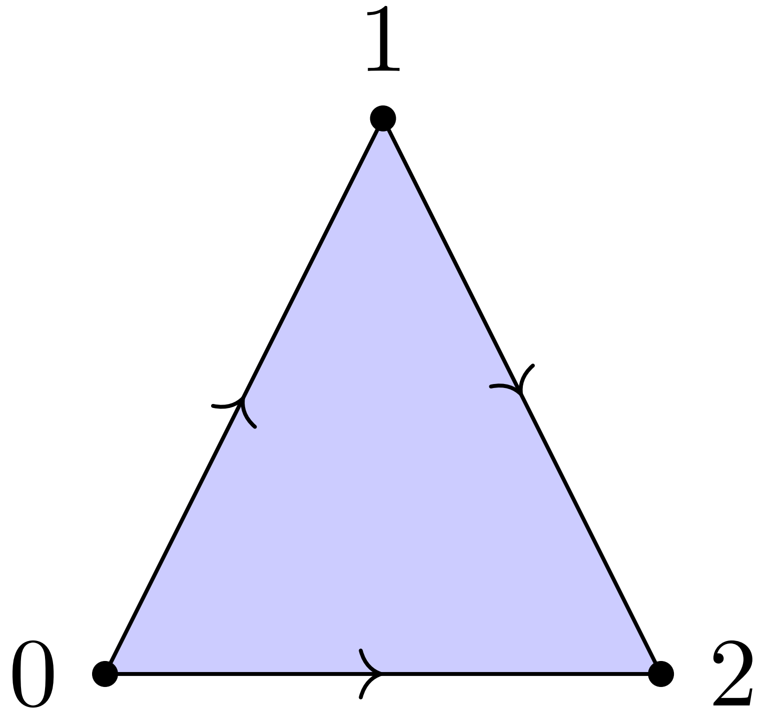
$$\mathbb{R}\mathrm{Hom}(c, d)$$

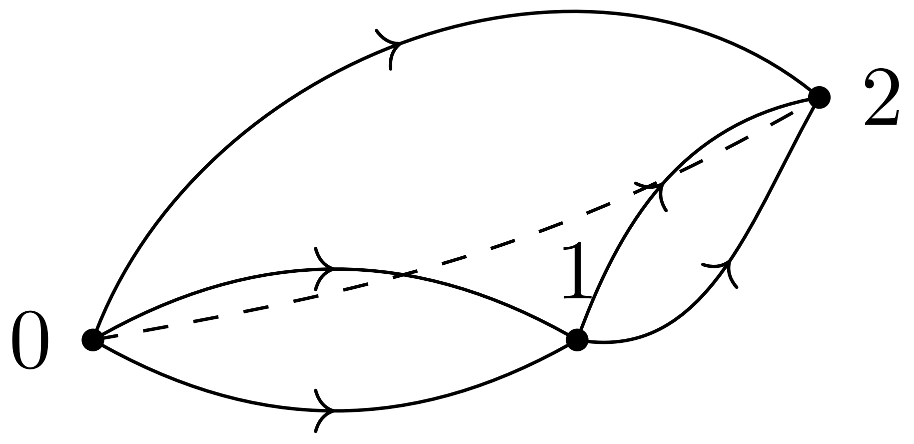
Homotopy Theorists are interested in the **model category** of spaces **Top**. However, nowadays, they use a combinatorial model of spaces called **simplicial sets**. The category **sSet** of simplicial sets has a model structure, and the two model categories are **Quillen equivalent**

$$\mathbf{sSet} \simeq \mathbf{Top}.$$

Simplicial sets are a lot like simplicial complexes, except their simplices are totally ordered, and there can be loops and multiple simplices with the same vertices.







I found that Matsushita (2016) had constructed a model structure on loop graphs

He constructed a functor $\mathrm{Sing} : \mathbf{Gr}_\ell \to \mathbf{sSet}$ defined as follows. Given a loop graph G, the set of n-simplices of $\mathrm{Sing}(G)$ is given by

$$\mathrm{Sing}(G)_n = \mathbf{Gr}_\ell(K_n, G),$$

the set of maps from the n-clique to G.

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the set of maps from the n-clique to G.

This creates a very large, unwieldy space! However, it is easy to see when we think of these as spaces that $\mathrm{Sing}(G)$ is weak equivalent to the clique complex

$$|\mathrm{C}\ell(G)| \simeq |\mathrm{Sing}(G)|.$$

Matsushita then **transfers** the homotopy theory from $\mathbf{sSet} \simeq \mathbf{Top}$ to $\mathbf{Gr}_\ell$. However, he actually needs to do something more subtle. He first shows that there exists another functor $A : \mathbf{sSet} \to \mathbf{Gr}_\ell$, and then shows that the following pair of functors

$$\begin{tikzcd} \mathbf{Gr}_\ell \arrow[rr, bend right, "\mathrm{Ex}^2\mathrm{Sing}"'] \arrow[rr, phantom, "\perp"] & & \mathbf{sSet} \arrow[ll, bend right, "A\,\mathrm{Sd}^2"'] \end{tikzcd}$$

allows you to transfer the homotopy theory. Here Sd^2 is the second barycentric subdivision, and Ex^2 is kind of like its opposite, called extension.

Why do you need to barycentric subdivide twice, i.e. use Sd^2?

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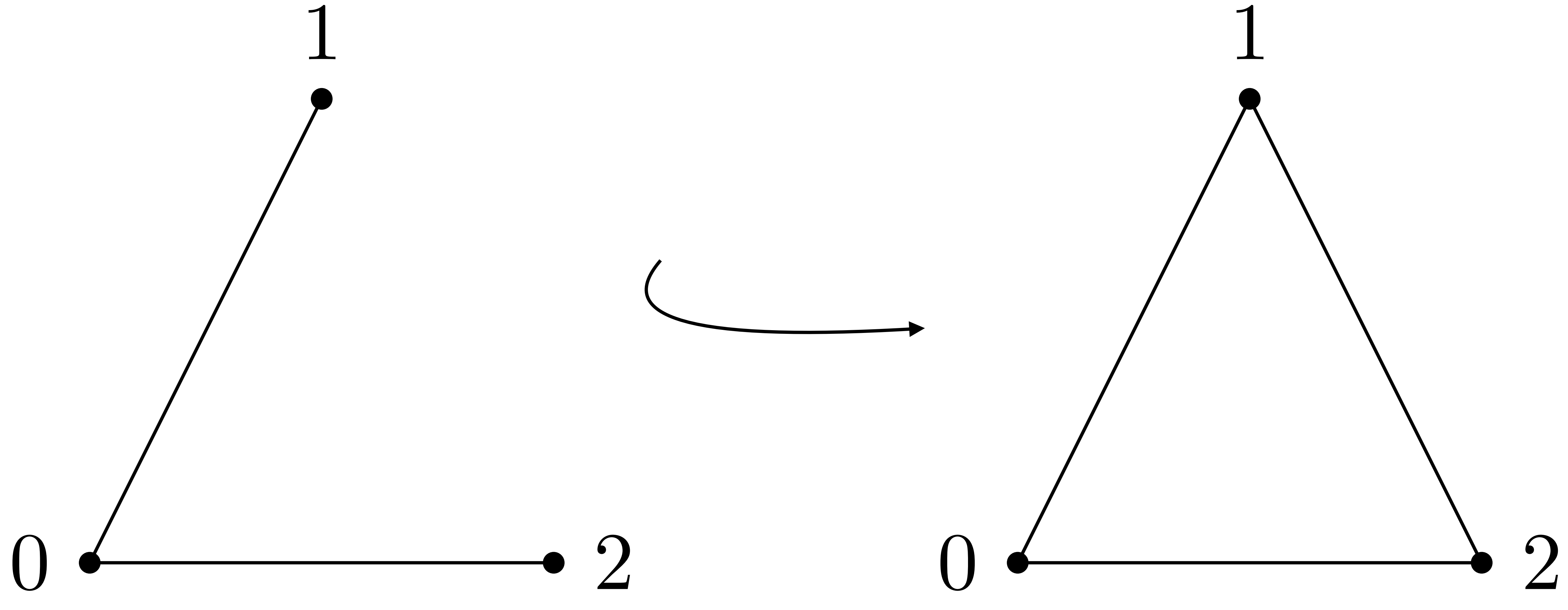
Given an inclusion of simplicial complexes $K \hookrightarrow L$, it turns out to be really useful for some neighborhood of K in L to collapse onto K. If (K, L) has this property, we call it an **NDR pair**.



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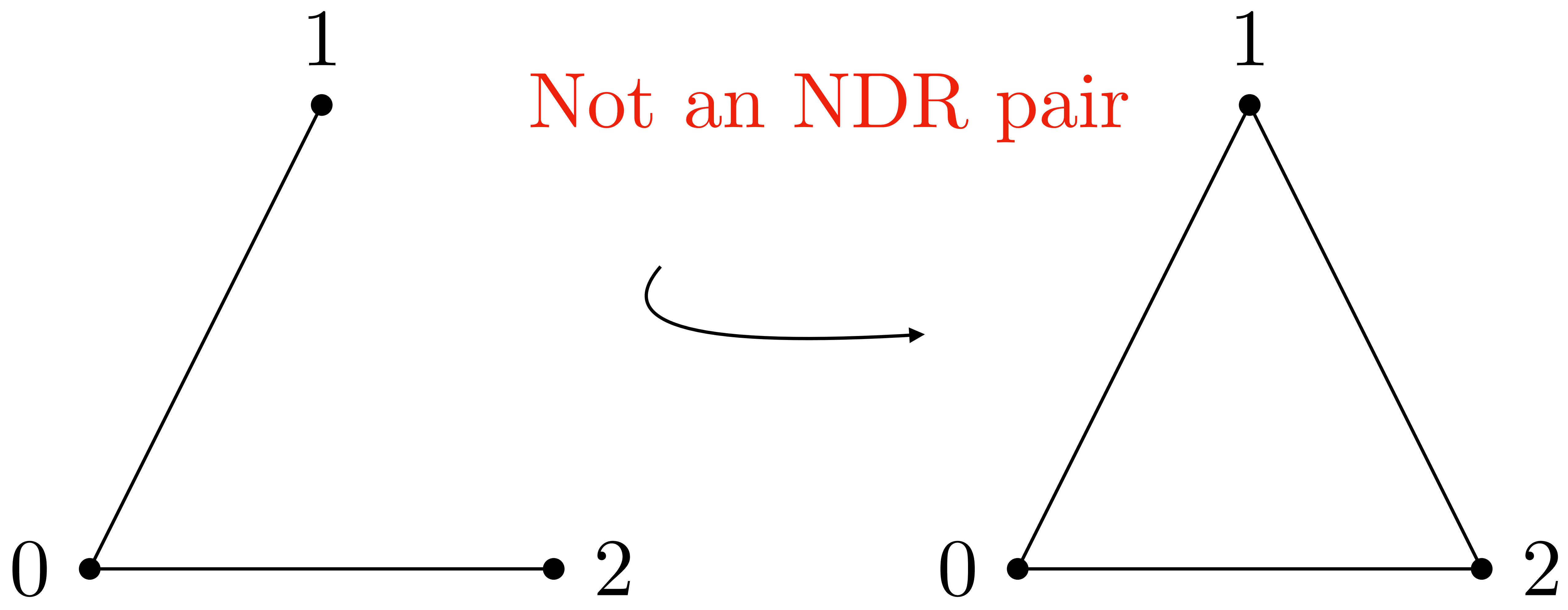
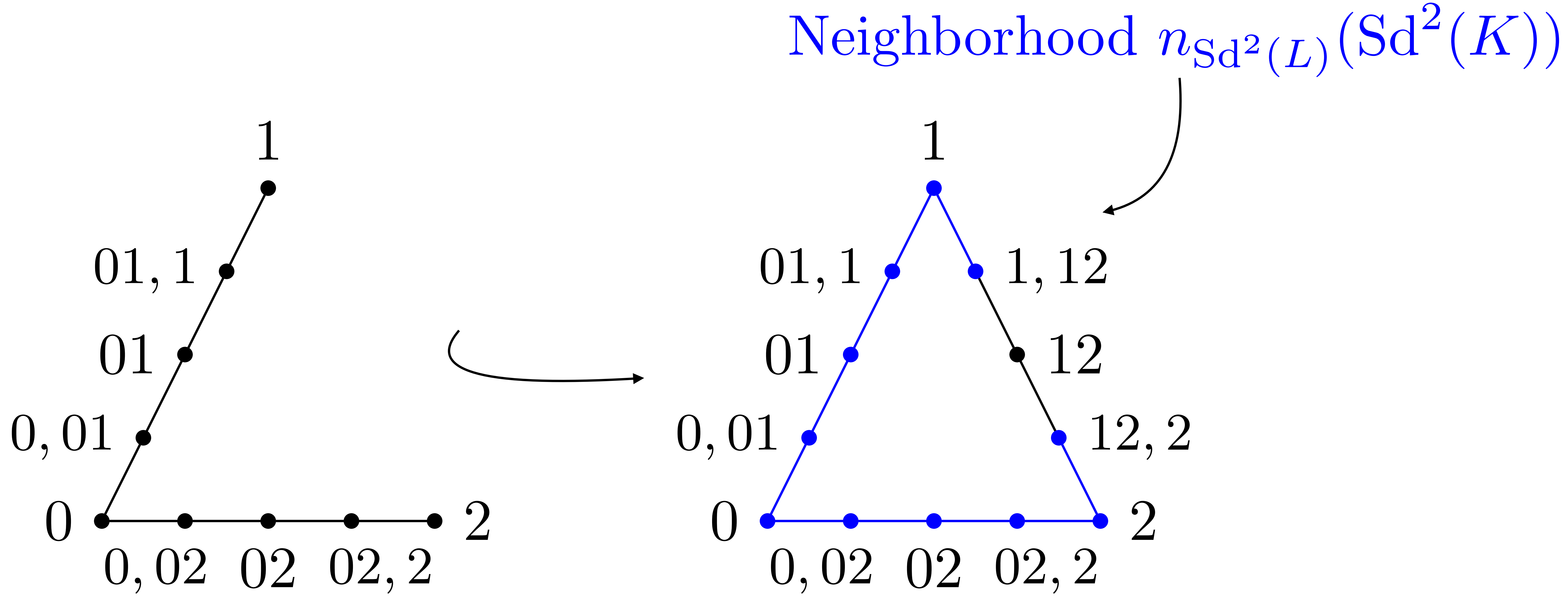


Amazing fact: Given any inclusion $K \hookrightarrow L$ of simplicial complexes, $\mathrm{Sd}^2(K) \to \mathrm{Sd}^2(L)$ is an NDR pair!



Theorem[M. 2026]:
Matsushita's construction factors through simplicial complexes and reflexive graphs. The classical model structure on **sSet** transfers to each of the following categories, and are all Quillen equivalent.

$$
\begin{tikzcd}
\mathbf{Gr}_\ell \arrow[r, bend right, "(-)^\circ"'] \arrow[r, phantom, "\perp"] & \mathbf{Gr}_r \arrow[l, bend right, "i_\ell"'] \arrow[r, bend right, "\mathrm{C}\ell"'] \arrow[r, phantom, "\perp"] & \mathbf{Cpx} \arrow[l, bend right, "(-)_{\leq 1}"'] \arrow[r, bend right, "\mathrm{Ex}^2\,\mathrm{Sing}"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet} \arrow[l, bend right, "\mathrm{Re}\,\mathrm{Sd}^2"']
\end{tikzcd}
$$

Furthermore, all of these model categories are proper, and there is a simple, concrete description of the functor A as the composite $i_\ell \circ (-)_{\leq 1} \circ \mathrm{Re}$.

What is $\mathrm{Re}: \mathbf{sSet} \to \mathbf{Cpx}$?

I hadn't seen this functor described in the literature, but it is very simple.

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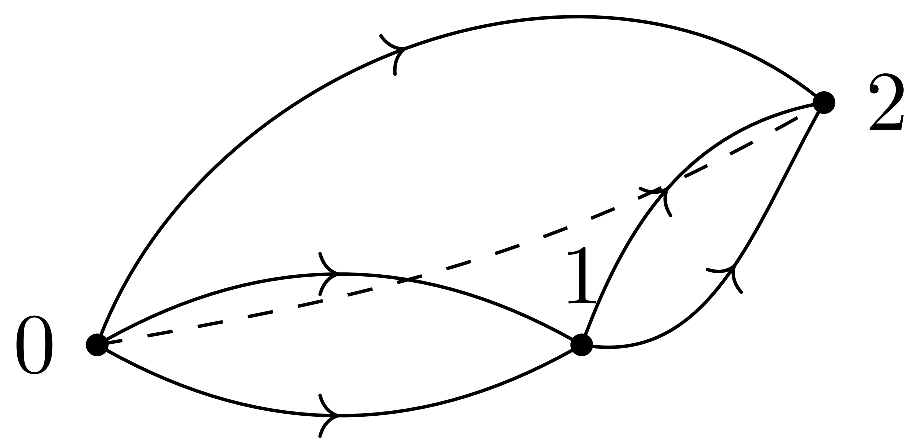
I hadn't seen this functor described in the literature, but it is very simple.

Can be described categorically as a coend

$$\mathrm{Re}(X) = \int^{n \in \Delta} X_n \times \underline{\Delta}^n$$

But much more simply: give the simplicial complex consisting of nondegenerate simplices with distinct vertices.



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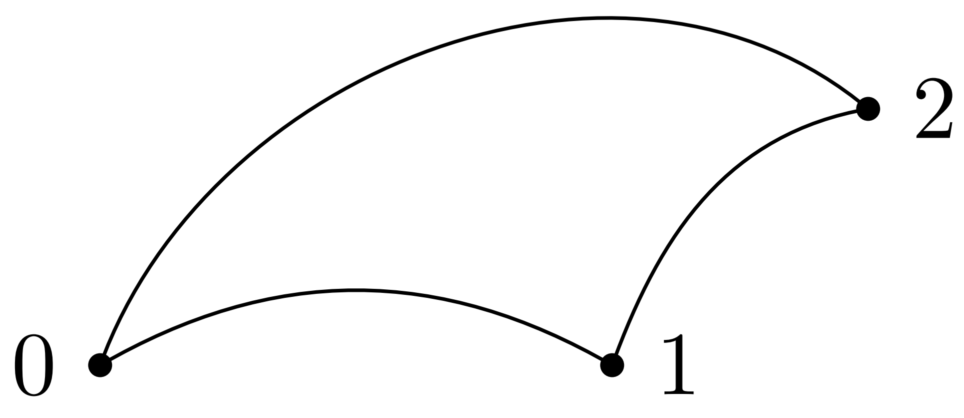

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It turned out that my original conjecture, that with this model structure

$$\mathbb{R}\mathrm{Hom}(G, H) \simeq \mathrm{C}\ell((H^G)^\circ))$$

was false :(

But this theorem stands on its own, even if my motivating conjecture was incorrect.

In fact, there is a corollary to this theorem that I find really interesting.

Let X, A, B be spaces such that $X = A \cup B$. The **Mayer-Vietoris Sequence** is an exact sequence which tells you the homology of X in terms of the homology of A, B and $A \cap B$.

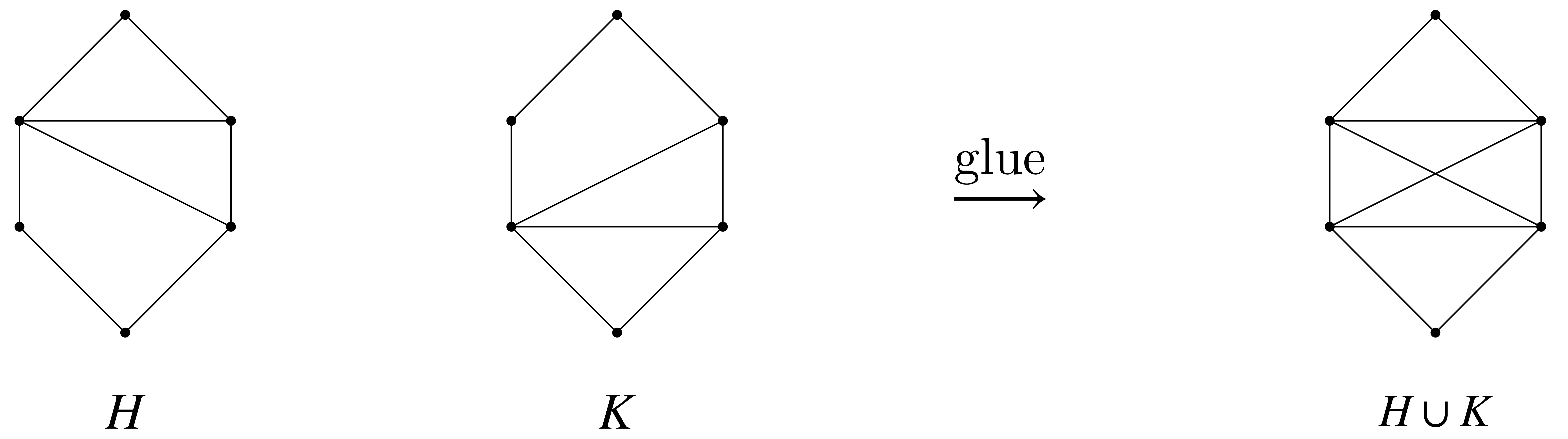
$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \ldots$$

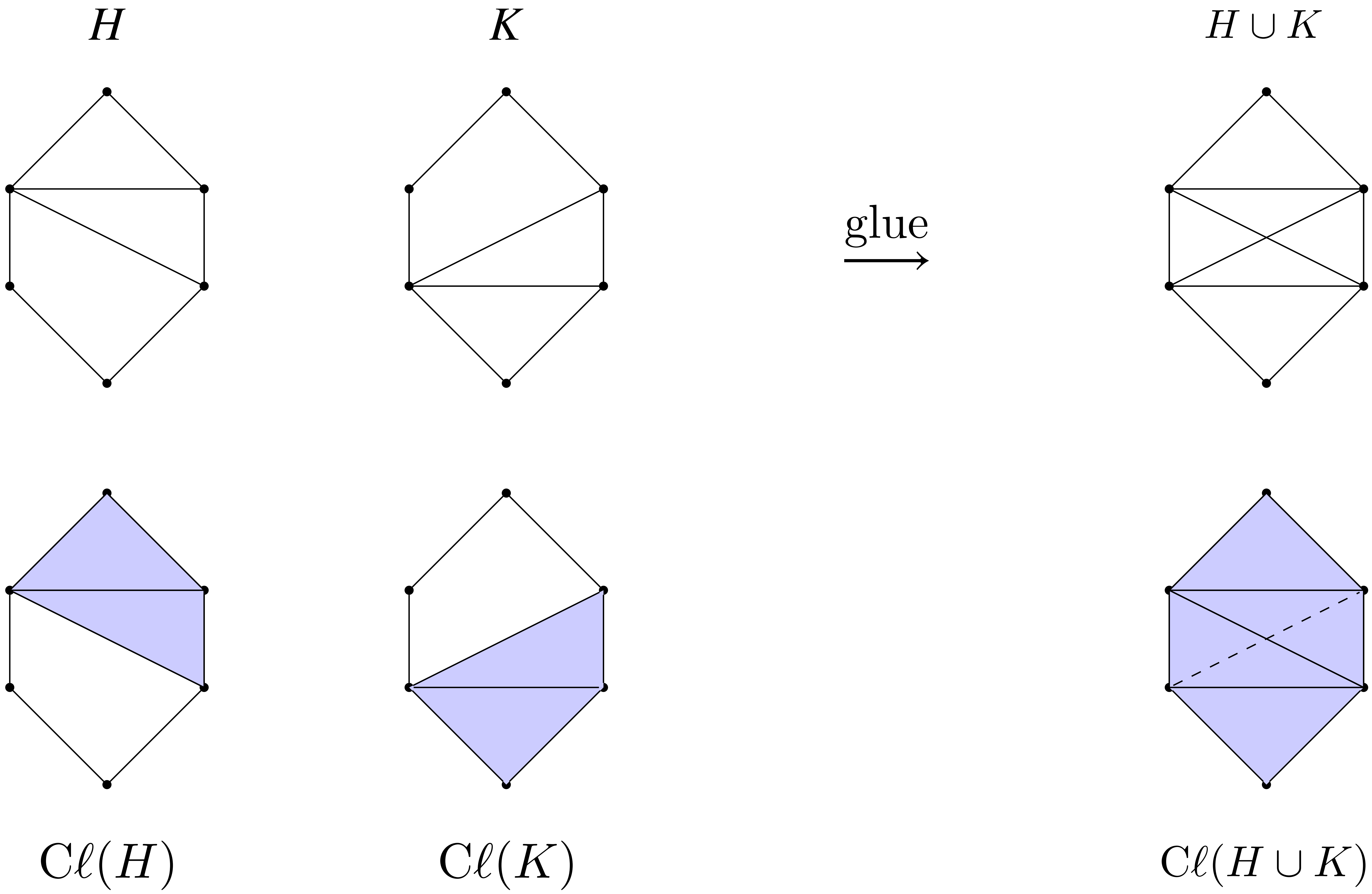
Let X, A, B be spaces such that $X = A \cup B$. The **Mayer-Vietoris Sequence** is an exact sequence which tells you the homology of X in terms of the homology of A, B and $A \cap B$.

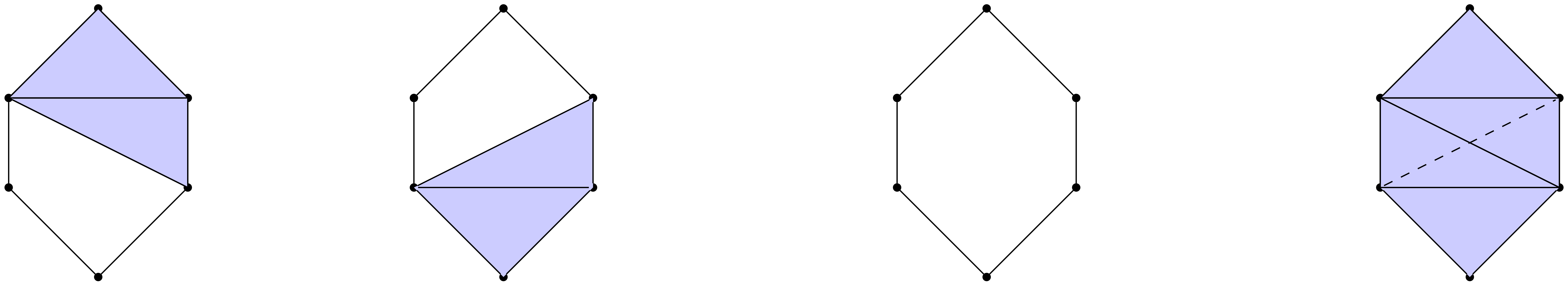
$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \dots$$

It would be nice if this was compatible with the clique complex.

Given graphs H and K, let's glue them together to get $H \cup K$. But now, if we compare $\mathrm{C}\ell(H)$ and $\mathrm{C}\ell(K)$ with $\mathrm{C}\ell(H \cup K)$, the Mayer-Vietoris sequence is not exact!

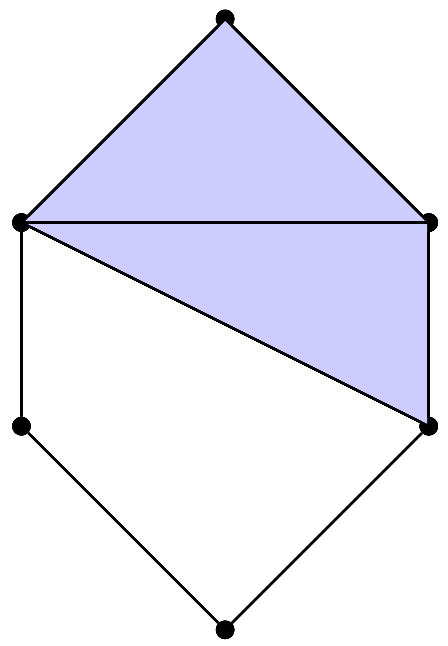



H
K
$H \cup K$
glue
$\mathrm{C}\ell(H)$
$\mathrm{C}\ell(K)$
$\mathrm{C}\ell(H \cup K)$

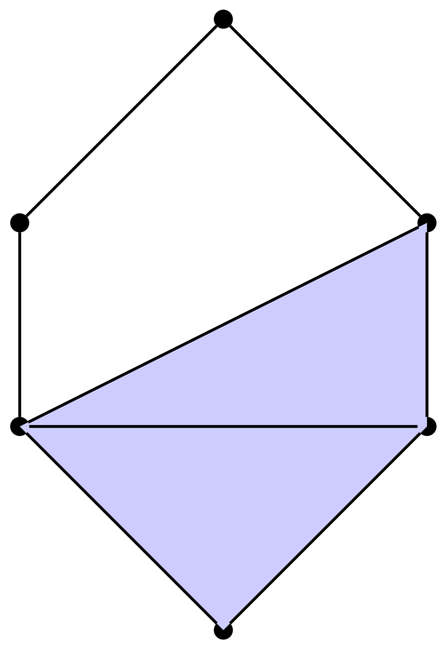


$\mathrm{C}\ell(H)$ $\quad$ $\mathrm{C}\ell(K)$ $\quad$ $\mathrm{C}\ell(H \cap K)$ $\quad$ $\mathrm{C}\ell(H \cup K)$

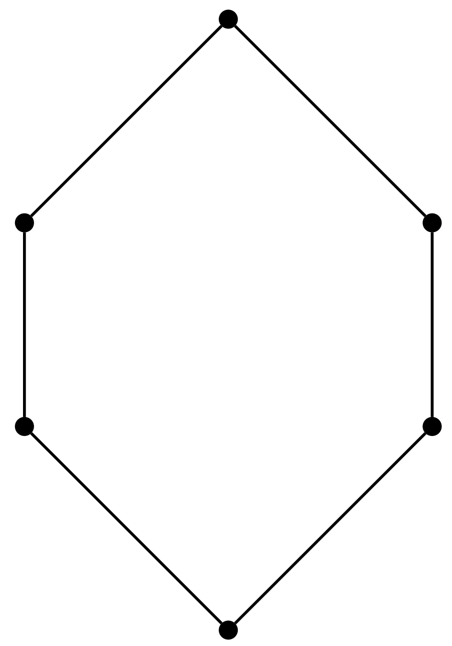
$$
\begin{tikzcd}
0 \arrow[r] & H_1(\mathrm{C}\ell(H \cap K)) \arrow[r] & H_1(\mathrm{C}\ell(H)) \oplus H_1(\mathrm{C}\ell(K)) \arrow[r] & H_1(\mathrm{C}\ell(H \cup K)) \arrow[dll, out=-10, in=170] & \\
 & \widetilde{H}_0(\mathrm{C}\ell(H \cap K)) \arrow[r] & \widetilde{H}_0(\mathrm{C}\ell(H)) \oplus \widetilde{H}_0(\mathrm{C}\ell(K)) \arrow[r] & \widetilde{H}_0(\mathrm{C}\ell(H \cup K)) \arrow[r] & 0
\end{tikzcd}
$$



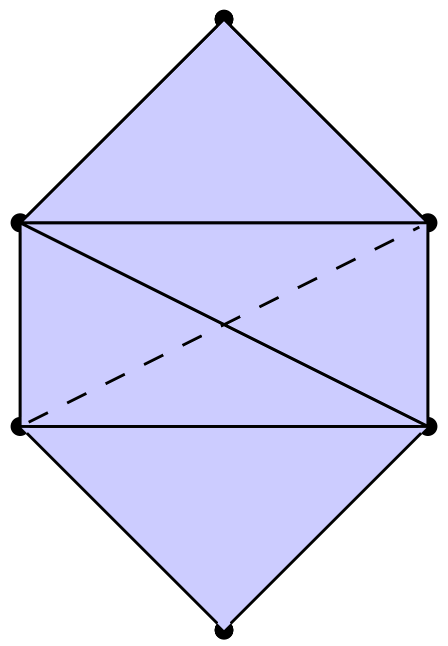
$\mathrm{C}\ell(H)$



$\mathrm{C}\ell(K)$



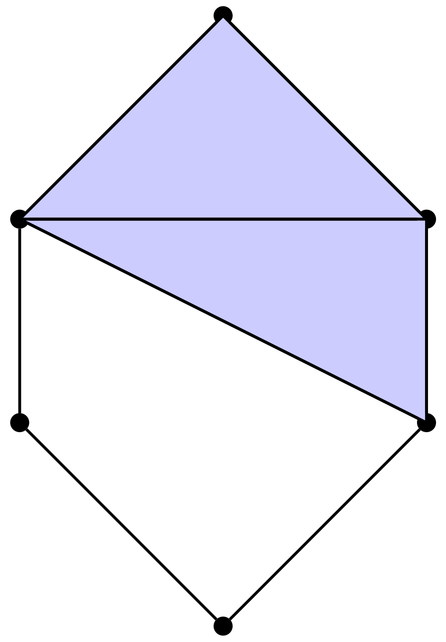
$\mathrm{C}\ell(H \cap K)$



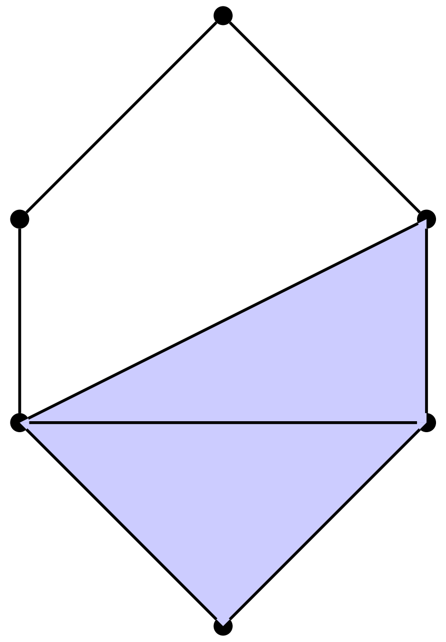
$\mathrm{C}\ell(H \cup K)$

$$\begin{tikzcd}
0 \arrow[r] & H_1(\mathrm{C}\ell(H \cap K)) \arrow[r] & H_1(\mathrm{C}\ell(H)) \oplus H_1(\mathrm{C}\ell(K)) \arrow[r] & H_1(\mathrm{C}\ell(H \cup K)) \arrow[dll, rounded corners] & \\
 & \widetilde{H}_0(\mathrm{C}\ell(H \cap K)) \arrow[r] & \widetilde{H}_0(\mathrm{C}\ell(H)) \oplus \widetilde{H}_0(\mathrm{C}\ell(K)) \arrow[r] & \widetilde{H}_0(\mathrm{C}\ell(H \cup K)) \arrow[r] & 0
\end{tikzcd}$$

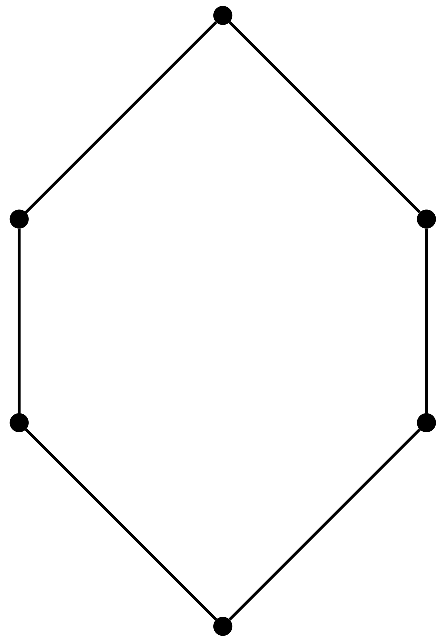
$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & 0 \arrow[dll, rounded corners] \\
 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$



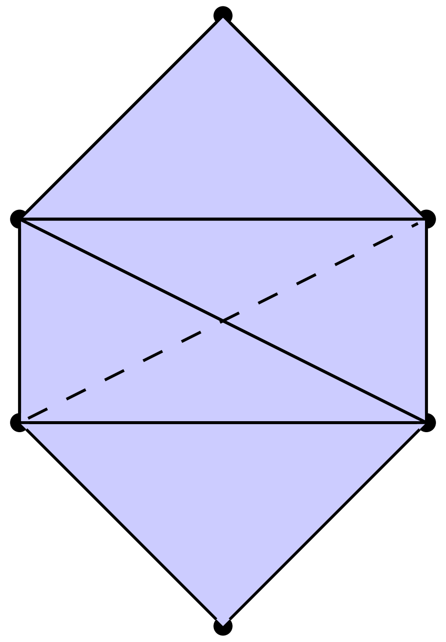
$\mathrm{C}\ell(H)$



$\mathrm{C}\ell(K)$



$\mathrm{C}\ell(H \cap K)$



$\mathrm{C}\ell(H \cup K)$

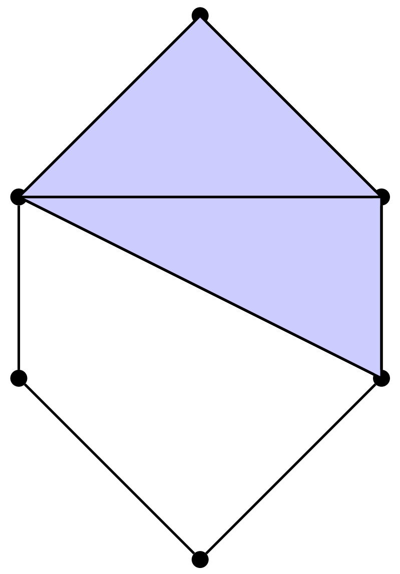
$$
\begin{aligned}
0 \longrightarrow H_1(\mathrm{C}\ell(H \cap K)) &\longrightarrow H_1(\mathrm{C}\ell(H)) \oplus H_1(\mathrm{C}\ell(K)) \longrightarrow H_1(\mathrm{C}\ell(H \cup K)) \longrightarrow \\
\longrightarrow \widetilde{H}_0(\mathrm{C}\ell(H \cap K)) &\longrightarrow \widetilde{H}_0(\mathrm{C}\ell(H)) \oplus \widetilde{H}_0(\mathrm{C}\ell(K)) \longrightarrow \widetilde{H}_0(\mathrm{C}\ell(H \cup K)) \longrightarrow 0
\end{aligned}
$$

$$
\begin{aligned}
0 \longrightarrow \mathbb{Z} &\longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \longrightarrow \\
\longrightarrow 0 &\longrightarrow 0 \longrightarrow 0 \longrightarrow 0
\end{aligned}
$$

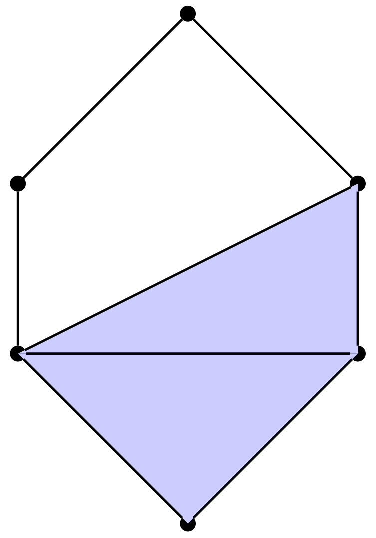
Not exact

The theorem before tells you how to get Mayer-Vietoris play nicely with clique complexes

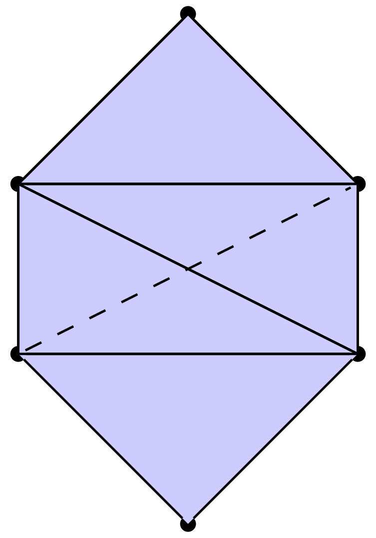
The key is double barycentric subdivision!



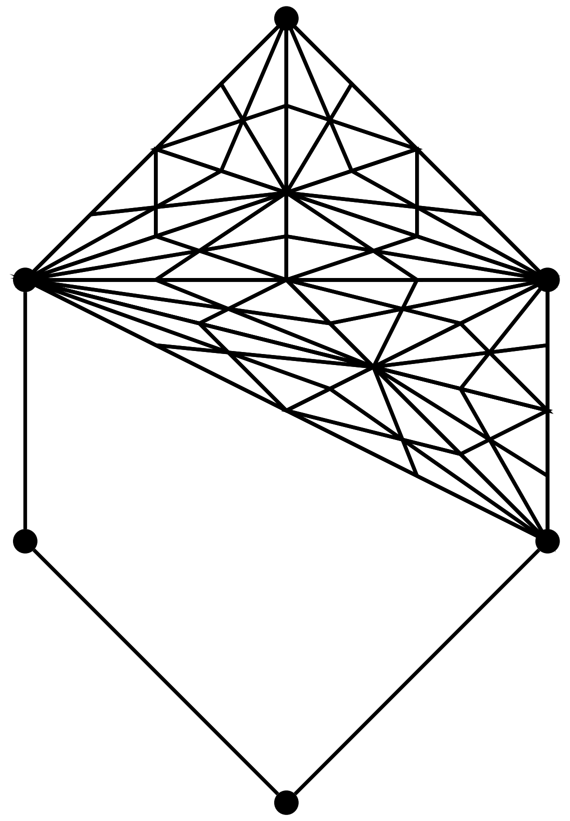
$\mathrm{C}\ell(H)$



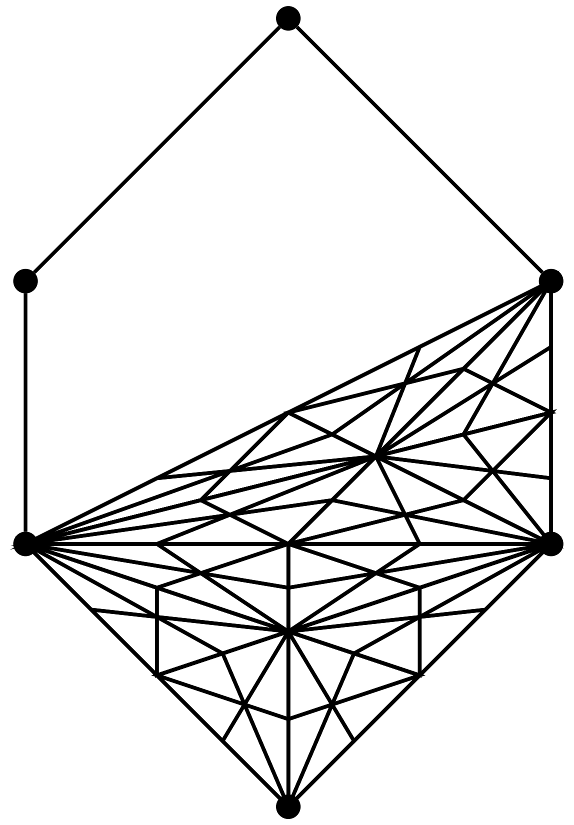
$\mathrm{C}\ell(K)$



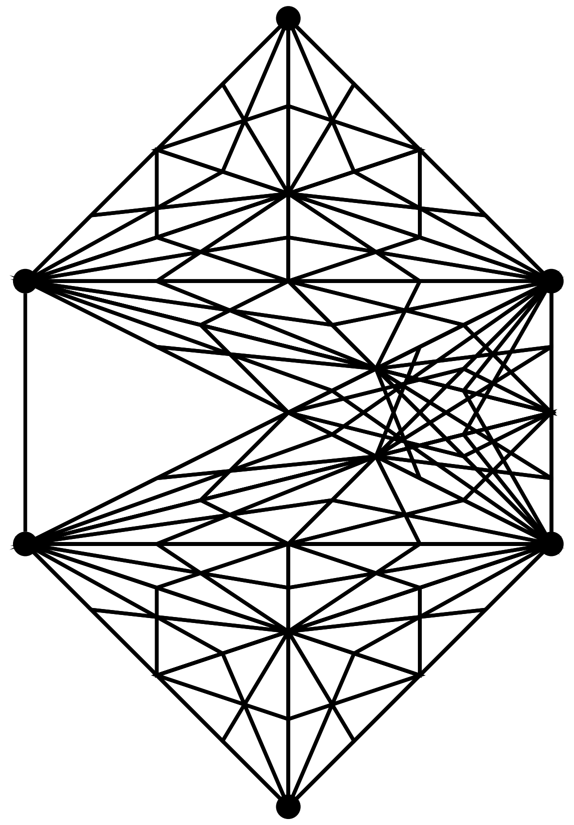
$\mathrm{C}\ell(G)$



$\widetilde{H} = (\mathrm{Sd}^2\, \mathrm{C}\ell(H))_{\leq 1}$

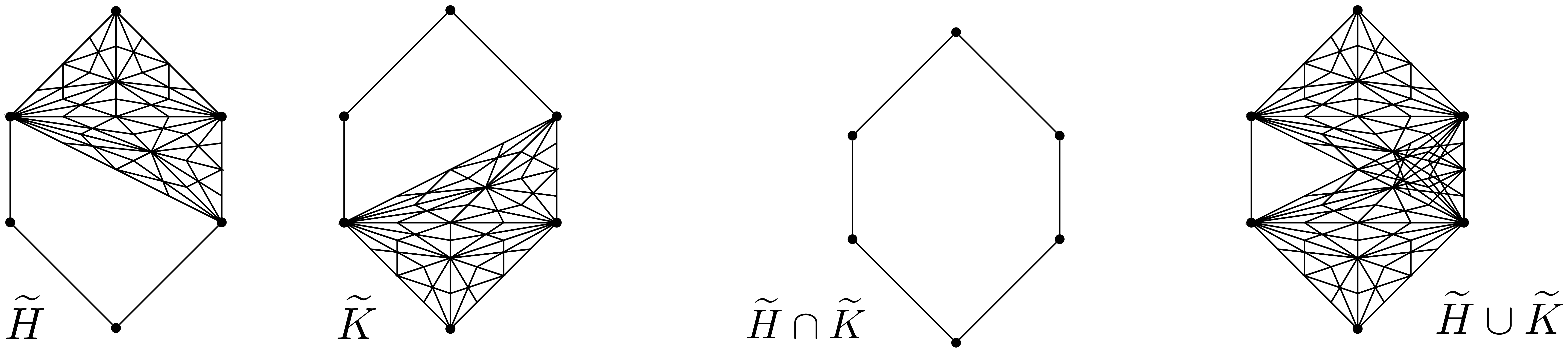


$\widetilde{K} = (\mathrm{Sd}^2\, \mathrm{C}\ell(K))_{\leq 1}$



$\widetilde{H} \cup \widetilde{K}$

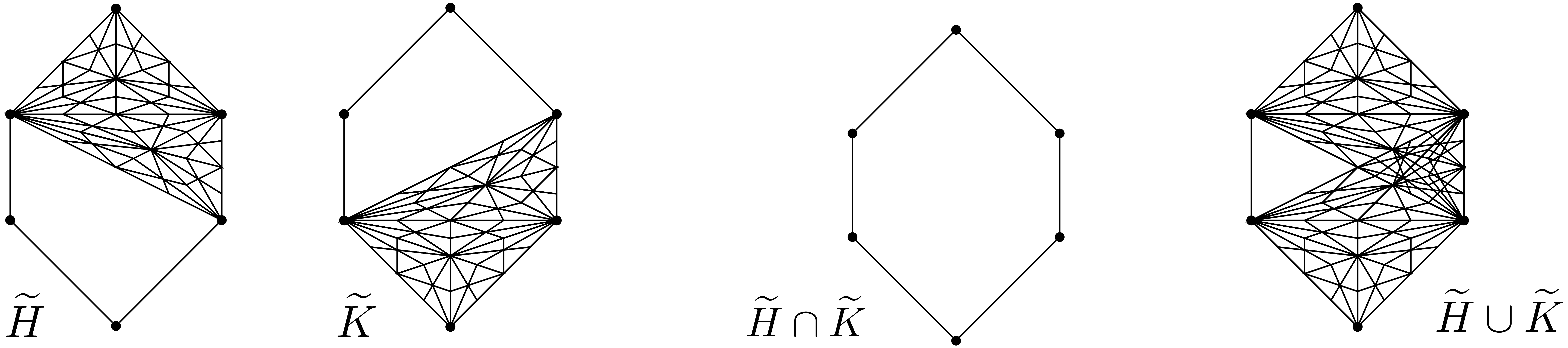
Now Mayer-Vietoris works correctly



$$0 \longrightarrow H_1(\mathrm{C}\ell(\widetilde{H} \cap \widetilde{K})) \longrightarrow H_1(\mathrm{C}\ell(\widetilde{H})) \oplus H_1(\mathrm{C}\ell(\widetilde{K})) \longrightarrow H_1(\mathrm{C}\ell(\widetilde{H} \cup \widetilde{K}))$$

$$\longrightarrow \widetilde{H}_0(\mathrm{C}\ell(\widetilde{H} \cap \widetilde{K})) \longrightarrow \widetilde{H}_0(\mathrm{C}\ell(\widetilde{H})) \oplus \widetilde{H}_0(\mathrm{C}\ell(\widetilde{K})) \longrightarrow \widetilde{H}_0(\mathrm{C}\ell(\widetilde{H} \cup \widetilde{K})) \longrightarrow 0$$

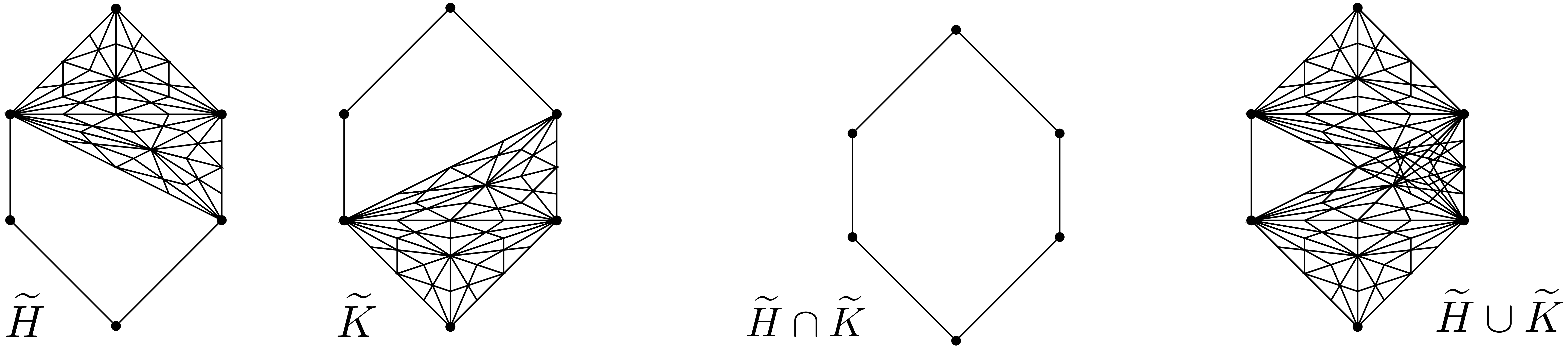
Now Mayer-Vietoris works correctly



$$
\begin{aligned}
0 \longrightarrow H_1(\mathrm{C}\ell(\widetilde{H} \cap \widetilde{K})) &\longrightarrow H_1(\mathrm{C}\ell(\widetilde{H})) \oplus H_1(\mathrm{C}\ell(\widetilde{K})) \longrightarrow H_1(\mathrm{C}\ell(\widetilde{H} \cup \widetilde{K})) \\
\longrightarrow \widetilde{H}_0(\mathrm{C}\ell(\widetilde{H} \cap \widetilde{K})) &\longrightarrow \widetilde{H}_0(\mathrm{C}\ell(\widetilde{H})) \oplus \widetilde{H}_0(\mathrm{C}\ell(\widetilde{K})) \longrightarrow \widetilde{H}_0(\mathrm{C}\ell(\widetilde{H} \cup \widetilde{K})) \longrightarrow 0
\end{aligned}
$$

$$
\begin{aligned}
0 \longrightarrow \mathbb{Z} &\longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \\
\longrightarrow 0 &\longrightarrow 0 \longrightarrow 0 \longrightarrow 0
\end{aligned}
$$

Now Mayer-Vietoris works correctly



$$
\begin{tikzcd}
0 \arrow[r] & H_1(\mathrm{C}\ell(\widetilde{H} \cap \widetilde{K})) \arrow[r] & H_1(\mathrm{C}\ell(\widetilde{H})) \oplus H_1(\mathrm{C}\ell(\widetilde{K})) \arrow[r] & H_1(\mathrm{C}\ell(\widetilde{H} \cup \widetilde{K})) \arrow[dll, out=-10, in=170] & \\
& \widetilde{H}_0(\mathrm{C}\ell(\widetilde{H} \cap \widetilde{K})) \arrow[r] & \widetilde{H}_0(\mathrm{C}\ell(\widetilde{H})) \oplus \widetilde{H}_0(\mathrm{C}\ell(\widetilde{K})) \arrow[r] & \widetilde{H}_0(\mathrm{C}\ell(\widetilde{H} \cup \widetilde{K})) \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dlll, out=-10, in=170] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

Exact✓

Summary:

- Lovász showed us that topology can inform combinatorics,
- Matsushita showed that there is a model structure on loop graphs,
- I extended Matsushita's result to simplicial complexes and reflexive graphs,
- I also showed that the model categories are proper, which gives a way of "repairing" Mayer-Vietoris.

Further questions:

- Can these results be extended to other kinds of spaces obtained from graphs:
  - Cubical complexes for A-homotopy Theory, path complexes for path homology?
- How do we compute arbitrary homotopy colimits of simplicial complexes?
- What other graph constructions play well with the clique complex functor?

Thank you for your patience!

Questions? Comments?

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