The Diffeological Čech-de Rham Obstruction

Monthly Global Diffeology Seminar

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Hello!

I’m Emilio Minichiello, I am an assistant professor at CUNY CityTech in New York City. Today I’m going to talk about my paper “The Diffeological Čech-de Rham Obstruction.”

It is about using categorical homotopy theory to study diffeological spaces.

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The Čech-de Rham Obstruction

For finite dimensional smooth manifolds, we have the

de Rham Theorem: If $M \in \mathbf{Man}$, and $k \geqslant 0$, then there is an isomorphism

$$H^k_{\mathrm{dR}}(M) \quad \cong \quad \check{H}^k(M, \mathbb{R}^\delta),$$

where the left hand is **de Rham cohomology**, and the right hand side is **Čech cohomology** with values in the discrete abelian group $\mathbb{R}^\delta$.

The Čech-de Rham Obstruction

How do we compute Čech cohomology?

From Bott and Tu [BT82, Section 10]:

Given a presheaf A of abelian groups on M, take a good open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M, and consider the following simplicial manifold

$$U_i \underset{d^1}{\overset{d^0}{\leftleftarrows}} U_i \cap U_j \; \begin{array}{c} \xleftarrow{d^0} \\ \xleftarrow{d^1} \\ \xleftarrow{d^2} \end{array} \; U_i \cap U_j \cap U_k \qquad \cdots$$

where the maps d^k are inclusions.

The Čech-de Rham Obstruction

Get a cosimplicial diagram of abelian groups

$$\prod_{i\in I} A(U_i) \underset{A(d^1)}{\overset{A(d^0)}{\rightrightarrows}} \prod_{i,j\in I} A(U_{ij}) \overset{A(d^0),\, A(d^1),\, A(d^2)}{\rightrightarrows\!\!\!\to} \prod_{i,j,k\in I} A(U_{ijk}) \qquad \cdots$$

We turn this into a cochain complex

$$\prod_{i\in I} A(U_i) \xrightarrow{\delta} \prod_{i,j\in I} A(U_{ij}) \xrightarrow{\delta} \prod_{i,j,k\in I} A(U_{ijk}) \qquad \cdots$$

by taking

$$\delta = \sum_{k=0}^{n} (-1)^k A(d^k).$$

The cohomology of

$$\check{C}(\mathcal{U}, A) := \prod_{i \in I} A(U_i) \xrightarrow{\delta} \prod_{i,j \in I} A(U_{ij}) \xrightarrow{\delta} \prod_{i,j,k \in I} A(U_{ijk}) \ldots$$

is the **Čech cohomology** of M, with values in A:

$$\check{H}^k(M, A).$$

The Čech-de Rham Obstruction

The de Rham theorem is very easy to prove once you know about spectral sequences. First you set up the Čech-de Rham double complex

$$\begin{array}{ccccc}
\prod_i \Omega^2(U_i) & \xrightarrow[\delta]{} & \prod_{i,j} \Omega^2(U_{ij}) & \xrightarrow[\delta]{} & \prod_{i,j,k} \Omega^2(U_{ijk}) \\
d\uparrow & & d\uparrow & & \uparrow d \\
\prod_i \Omega^1(U_i) & \xrightarrow[\delta]{} & \prod_{i,j} \Omega^1(U_{ij}) & \xrightarrow[\delta]{} & \prod_{i,j,k} \Omega^1(U_{ijk}) \\
d\uparrow & & \uparrow d & & \uparrow d \\
\prod_i \Omega^0(U_i) & \xrightarrow[\delta]{} & \prod_{i,j} \Omega^0(U_{ij}) & \xrightarrow[\delta]{} & \prod_{i,j,k} \Omega^0(U_{ijk})
\end{array}$$

If we take cohomology in the vertical direction, we end up with

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \\ \prod_i \mathbb{R}^\delta(U_i) & \xrightarrow{\delta} & \prod_{i,j} \mathbb{R}^\delta(U_{ij}) & \xrightarrow{\delta} & \prod_{i,j,k} \mathbb{R}^\delta(U_{ijk}) \end{array}$$

Because $\mathcal{U}$ was a good cover, so the de Rham cohomology of each $U_{i_0 \dots i_n}$ vanishes except in degree 0.

Here $\mathbb{R}^\delta(U_i)$ is the set of smooth maps from U_i to $\mathbb{R}^\delta$, which is $\mathbb{R}^\delta$ as a set. Every map is constant as $\mathbb{R}^\delta$ is discrete and U_i is connected.

Now taking cohomology horizontally we end up with

0	0	0
0	0	0
$\check{H}^0(M, \mathbb{R}^\delta)$	$\check{H}^1(M, \mathbb{R}^\delta)$	$\check{H}^2(M, \mathbb{R}^\delta)$

The Čech-de Rham Obstruction

Now lets take cohomology the other way, again starting with the double complex

$$\begin{array}{ccccc}
\prod_i \Omega^2(U_i) & \xrightarrow[\delta]{} & \prod_{i,j} \Omega^2(U_{ij}) & \xrightarrow[\delta]{} & \prod_{i,j,k} \Omega^2(U_{ijk}) \\
d\uparrow & & d\uparrow & & d\uparrow \\
\prod_i \Omega^1(U_i) & \xrightarrow[\delta]{} & \prod_{i,j} \Omega^1(U_{ij}) & \xrightarrow[\delta]{} & \prod_{i,j,k} \Omega^1(U_{ijk}) \\
d\uparrow & & d\uparrow & & d\uparrow \\
\prod_i \Omega^0(U_i) & \xrightarrow[\delta]{} & \prod_{i,j} \Omega^0(U_{ij}) & \xrightarrow[\delta]{} & \prod_{i,j,k} \Omega^0(U_{ijk})
\end{array}$$

The Čech-de Rham Obstruction

Now taking cohomology horizontally first, we end up with

$$\begin{array}{ccc} \Omega^2(M) & 0 & 0 \\ d\uparrow & \uparrow & \uparrow \\ \Omega^1(M) & 0 & 0 \\ d\uparrow & \uparrow & \uparrow \\ \Omega^0(M) & 0 & 0 \end{array}$$

This is because M has **partitions of unity**. This is a vital point.

Then taking cohomology vertically, we obtain de Rham cohomology.

$$\begin{array}{ccc} H^2_{\mathrm{dR}}(M) & 0 & 0 \\ \\ H^1_{\mathrm{dR}}(M) & 0 & 0 \\ \\ H^0_{\mathrm{dR}}(M) & 0 & 0 \end{array}$$

The Čech-de Rham Obstruction

Thus both spectral sequences one gets from the double complex collapse at the E_2 page, and we end up with the de Rham theorem

$$H^k_{\mathrm{dR}}(M) \cong \check{H}^k(M, \mathbb{R}^\delta).$$

What about for diffeological spaces?

The Čech-de Rham Obstruction

In 1988, Patrick Iglesias-Zemmour showed that this isomorphism does not hold for all diffeological spaces. In particular it does not hold for the irrational torus!

This was written in a preprint "Bi-complexe cohomologique des espaces differéntiables" in French and was never published.

In 2020, PIZ rewrote and revised this paper as "Čech-De Rham Bicomplex in Diffeology", [Igl24], it appeared in the Israel Journal of Mathematics in 2024.

Let's discuss a bit about what Patrick did in this paper.

The Čech-de Rham Obstruction

The first observation we want to make is that we don't want to use open covers to study diffeological spaces.

For example, the underlying D-topology of the irrational torus T_α is $\{\varnothing, T_\alpha\}$. So there's no interesting information we can obtain from the open covers of T_α.

PIZ instead uses what he calls the **Gauge monoid** of a diffeological space.

The Čech-de Rham Obstruction

Given a diffeological space X, let

$$B = \sum_{p \in \mathbf{Plot}(X)} U_p$$

be the coproduct of all the domains of all the round plots $p : U_p \to X$ of X. There's a map $\pi : B \to X$ given componentwise by the plots.

We define M, the Gauge monoid of X, to be the set of maps

$$\begin{array}{ccc} B & \xrightarrow{\ f\ } & B \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

equipped with the sub-diffeology of the functional diffeology on $C^\infty(B, B)$.

Now M is a diffeological monoid using composition of functions. It acts on B by moving points around to different plots. We then get a simplicial diffeological space

$$B \; \substack{\longleftarrow \\ \longleftarrow} \; B \times M \; \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \; B \times M \times M \qquad \cdots$$

which is the Bar construction of M acting on B.

Now if A is a diffeological abelian group, like $\mathbb{R}^\delta$, then we can map each piece into it, giving a cosimplicial abelian group

$$A^B \; \substack{\longrightarrow \\ \longrightarrow} \; A^{B\times M} \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; A^{B\times M\times M} \qquad \cdots$$

The Čech-de Rham Obstruction

From here we use the same trick[1] to obtain a cochain complex

$$A^B \xrightarrow{\delta} A^{B\times M} \xrightarrow{\delta} A^{B\times M\times M} \qquad \dots$$

The cohomology of this cochain complex is then PIZ's version of Čech cohomology

$$\check{H}^k_{\mathrm{PIZ}}(X, A).$$

We'll call this PIZ cohomology.

[1]This "trick" is called the co-Dold Kan correspondence

The Čech-de Rham Obstruction

Let $K \subseteq \mathbb{R}$ be a diffeologically discrete subgroup of $\mathbb{R}$. Then let $T_K = \mathbb{R}/K$. When $K = \mathbb{Z} + \alpha\mathbb{Z}$, we have the usual irrational torus T_α. Using the construction of PIZ cohomology, PIZ proves the following wonderful result:

Theorem[Igl24]:

$$\check{H}^k_{\mathrm{PIZ}}(T_K, \mathbb{R}^\delta) \cong H^k_{\mathrm{grp}}(K, \mathbb{R}),$$

where $H^k_{\mathrm{grp}}(K, \mathbb{R})$ is the group cohomology of K with coefficients in $\mathbb{R}$.

The proof of this is a little complicated, and quite computational.

The Čech-de Rham Obstruction

PIZ also considers the following double complex. Let $B//M$ be the simplicial diffeological space coming from the gauge monoid of a diffeological space X. So $B//M_k = B \times M^{\times k}$. Then we get an analogue of the Čech-de Rham double complex:

$$\begin{array}{ccccc}
\Omega^2(B//M_0) & \xrightarrow[\delta]{} & \Omega^2(B//M_1) & \xrightarrow[\delta]{} & \Omega^2(B//M_2) \\
d\uparrow & & d\uparrow & & \uparrow d \\
\Omega^1(B//M_0) & \xrightarrow[\delta]{} & \Omega^1(B//M_1) & \xrightarrow[\delta]{} & \Omega^1(B//M_2) \\
d\uparrow & & \uparrow d & & \uparrow d \\
\Omega^0(B//M_0) & \xrightarrow[\delta]{} & \Omega^0(B//M_1) & \xrightarrow[\delta]{} & \Omega^0(B//M_2)
\end{array}$$

The Čech-de Rham Obstruction

Taking cohomology vertically and then horizontally gives PIZ cohomology

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \check{H}^0_{\mathrm{PIZ}}(X, \mathbb{R}^\delta) & \check{H}^1_{\mathrm{PIZ}}(X, \mathbb{R}^\delta) & \check{H}^2_{\mathrm{PIZ}}(X, \mathbb{R}^\delta) \end{array}$$

The Čech-de Rham Obstruction

But now the story changes! Taking cohomology horizontally and then vertically no longer collapses. Because diffeological spaces in general do not have partitions of unity! We end up with a bunch of messy terms.

$$\begin{array}{ccc} H^2_{\mathrm{dR}}(X) & {}^dE_2^{1,2} & {}^dE_2^{2,2} \\ \\ H^1_{\mathrm{dR}}(X) & {}^dE_2^{1,1} & {}^dE_2^{2,1} \\ \\ H^0_{\mathrm{dR}}(X) & {}^dE_2^{1,0} & {}^dE_2^{2,0} \end{array}$$

So we have two spectral sequences converging to the total cohomology of the bicomplex

$$ {}^{\delta}E_{*}^{p,q} \Rightarrow H_{\mathrm{tot}}, \qquad {}^{d}E_{*}^{p,q} \Rightarrow H_{\mathrm{tot}}. $$

But ${}^{\delta}E_{*}^{p,q}$ collapses at the E_2 page, and gives PIZ cohomology, i.e. ${}^{\delta}E_{2}^{0,q} = \check{H}^{q}_{\mathrm{PIZ}}(X, \mathbb{R}^{\delta})$. In other words, the total cohomology of the bicomplex is precisely PIZ cohomology, $H^{q}_{\mathrm{tot}} = \check{H}^{q}_{\mathrm{PIZ}}(X, \mathbb{R}^{\delta})$.

We can take advantage of this by considering the **five-term exact sequence** of the other spectral sequence ${}^{d}E_{*}^{p,q}$.

Five-Term Exact Sequence: Suppose that ${}^{d}E_{*}^{p,q} \Rightarrow H$ is a spectral sequence converging to the cohomology H. Then we have an exact sequence

$$ 0 \to {}^{d}E_{2}^{0,1} \to H^{1} \to {}^{d}E_{2}^{1,0} \to {}^{d}E_{2}^{2,0} \to H^{2}. $$

From this, PIZ's result follows:

Theorem[Igl24]: Given a diffeological space X, we have the following exact sequence

$$0 \to H^1_{\mathrm{dR}}(X) \to \check{H}^1_{\mathrm{PIZ}}(X, \mathbb{R}^\delta) \to {}^dE_2^{1,0} \to H^2_{\mathrm{dR}}(X) \to \check{H}^2_{\mathrm{PIZ}}(X, \mathbb{R}^\delta)$$

PIZ was also able to identify the middle term:

$${}^dE_2^{1,0} \cong \check{H}^1_{\mathrm{conn}}(X, \mathbb{R}),$$

this is the group of isomorphism classes of $\mathbb{R}$-principal bundles on X that admit a connection 1-form.

The Čech-de Rham Obstruction

Note that the above theorem only holds for degree 1. There is no obvious way to extend this to an exact sequence in every degree using spectral sequences.

We turn now to a very different way of thinking about diffeological spaces, which will give us access to new tools to study their cohomology.

Diffeological spaces as simplicial presheaves

Diffeological spaces as simplicial presheaves

In 2008, John Baez and his student Alexander Hoffnung proved a wonderful theorem:

Theorem[BH11]: The category of diffeological spaces is equivalent to the category of concrete sheaves on the category of open subsets of cartesian spaces:

$$\mathbf{Diff} \cong \mathbf{ConSh}(\mathrm{Open}).$$

This result "explains" why the category of diffeological spaces is so nice, it also proves exactly how nice it is: **Diff** is a **quasitopos**.

The idea is that if X is a diffeological space and U is a cartesian space, then we let $X(U) = \{p : U \to X \,|\, p \text{ is a plot of } X\}$.

Diffeological spaces as simplicial presheaves

In my previous paper [Min24a], I took advantage of this theorem to think of diffeological spaces as certain kinds of **sheaves of spaces**, rather than sheaves of sets.

First, lets note that there's an equivalence

$$\mathbf{ConSh}(\mathrm{Open}) \cong \mathbf{ConSh}(\mathrm{Cart})$$

where Cart is the category of cartesian spaces. This is a technical convenience.

Then lets consider the category $\mathbf{sPre}(\mathrm{Cart})$. The objects are functors $X : \mathrm{Cart}^{\mathrm{op}} \to \mathbf{sSet}$. We call this the category of **simplicial presheaves**.

Diffeological spaces as simplicial presheaves

The category **sPre**(Cart) can be equipped with a really nice model structure, called the **Čech projective model structure**. We let $\check{\mathbb{H}}$ denote this category equipped with the Čech projective model structure.

We can think of this category as a big box, where all manifolds, diffeological spaces, and all kinds of crazy abstract mathematical objects live.

I like to think of objects in $\check{\mathbb{H}}$ as things with a simplicial "direction" and a smooth "direction".

The weak equivalences in $\check{\mathbb{H}}$ include the objectwise weak equivalences, and the fibrant objects are called **∞-stacks**. These are our main objects of interest. They will serve as coefficient objects for cohomology.

The category $\check{\mathbb{H}}$ is what is sometimes called a **model topos** or an **∞-topos**.[2]

[2] If you don't mind being a little imprecise. Technically it is only a model topos, but it presents an ∞-topos, and conversely every ∞-topos can be presented by a model topos.

Very briefly: $\check{\mathbb{H}}$ is a category where for every pair of objects $X, A \in \check{\mathbb{H}}$, there is a space

$$\mathbb{R}\check{\mathbb{H}}(X, A)$$

called the **derived mapping space**.

Taking π_0 of this space gives us a notion of cohomology:

$$\check{H}^0_\infty(X, A) := \pi_0 \mathbb{R}\check{\mathbb{H}}(X, A),$$

of X with values in A.

Thus you can think:

∞-topos $\simeq$ Nice framework for defining and manipulating cohomology.

Diffeological spaces as simplicial presheaves

We can consider the following inclusions:

$$\mathbf{Sh}(\mathrm{Cart}) \hookrightarrow \mathbf{Pre}(\mathrm{Cart}) \xhookrightarrow{{}^{c}(-)} \mathbf{sPre}(\mathrm{Cart}),$$

where for the last inclusion, if X is a presheaf on Cart, then we think of ${}^{c}X$ as the simplicial presheaf where if $U \in \mathrm{Cart}$, then ${}^{c}X(U)$ is the simplicial set where all the face and degeneracy maps are the identity.

So we can think of any diffeological space X as a functor $X : \mathrm{Cart}^{\mathrm{op}} \to \mathbf{sSet}$ in a really trivial way, first use the Baez-Hoffnung Theorem to identify X with a functor $X : \mathrm{Cart}^{\mathrm{op}} \to \mathbf{Set}$ and then just include $\mathbf{Set} \hookrightarrow \mathbf{sSet}$.

Diffeological spaces as simplicial presheaves

It might seem like we haven't done anything, and we really haven't. However, $\check{\mathbb{H}}$ comes equipped with a functor $Q : \check{\mathbb{H}} \to \check{\mathbb{H}}$, defined by Dugger [Dug01], called **cofibrant replacement**. If we apply this to a diffeological space X we obtain a really interesting simplicial presheaf:

$$QX \cong \left(B \leftleftarrows \sum_{U_{p_1} \xrightarrow{f_0} U_{p_0}} U_{p_1} \;\overleftarrow{\overleftarrow{\overleftarrow{}}}\; \sum_{U_{p_2} \xrightarrow{f_1} U_{p_1} \xrightarrow{f_0} U_{p_0}} U_{p_2} \quad \cdots \right)$$

where $B = \sum_{p_0 : U_{p_0} \to X} U_{p_0}$ is the nebula of X.

Diffeological spaces as simplicial presheaves

Now we can compute the ∞-**stack cohomology** of a diffeological space X. If A is an ∞-stack, then

$$\check{H}^0_\infty(X, A) := \pi_0 \mathbb{R}\check{\mathbb{H}}(X, A) = \pi_0 \underline{\check{\mathbb{H}}}(QX, A),$$

where $\underline{\check{\mathbb{H}}}(QX, A)$ is the simplicially-enriched Hom of **sPre**(Cart).

The higher ∞-stack cohomology of X is only defined for certain kinds of coefficient ∞-stacks A. There's an operation called delooping, and **if it exists**, the k-fold delooping of A is denoted $\mathbf{B}^k A$. So we have

$$\check{H}^k_\infty(X, A) := \pi_0 \mathbb{R}\check{\mathbb{H}}(X, \mathbf{B}^k A).$$

Diffeological spaces as simplicial presheaves

You can think of QX as a different way of "resolving" X. PIZ had a different "resolution," namely the gauge monoid $B//M$.

In fact, every time one has a "resolution" for diffeological spaces, we obtain a different notion of Čech cohomology! In [Min24a], I considered the two resolutions from before and another resolution from [KWW21], obtaining the following commutative diagram comparing different notions of Čech cohomology. Suppose that A is a diffeological abelian group:

$$\begin{array}{ccc} \check{H}^k_\infty(X,\mathbb{R}) & \underset{\text{res}*}{\overset{\delta*}{\rightleftarrows}} & H^k_{\mathrm{PIZ}}(X,A) \\ & {}_{(q\delta)*}\nwarrow \quad \nearrow_{q*} & \\ & \check{H}^k_{\mathrm{KWW}}(X,A) & \end{array}$$

Diffeological spaces as simplicial presheaves

$$
\begin{array}{ccc}
\check{H}^k_\infty(X,\mathbb{R}) & \overset{\delta^*}{\underset{\mathrm{res}^*}{\rightleftarrows}} & H^k_{\mathrm{PIZ}}(X,A) \\
& {\scriptstyle (q\delta)^*} \nwarrow \quad \nearrow {\scriptstyle q^*} & \\
& \check{H}^k_{\mathrm{KWW}}(X,A) &
\end{array}
$$

Open Question: Are all of these cohomologies isomorphic in all degrees? What about the diffeological Čech cohomology of [Ahm23]?

The Shape of a Diffeological Space

The Shape of a Diffeological Space

There is a very important functor

$$\int : \check{\mathbb{H}} \to \mathbf{sSet},$$

called the **shape** functor. It meshes together the simplicial and smooth direction of a simplicial presheaf, and spits out a homotopy type that takes both into account.

For example, if M is a manifold, then

$$\textstyle\int M \simeq Sing(M).$$

The shape functor is much beloved by higher differential geometers [BNV13], [Car15], [Bun22], [Sch13], [Clo23], [Pav22].

Lemma[Min24b]: If X is a diffeological space, then

$$\int X \simeq Sing_D(X) \simeq NPlot(X),$$

where $Sing_D(X)$ is the usual difffeological singular complex functor [Pav22], [CW14]:

$$Sing_D(X)_k := \mathbf{Diff}(\Delta_a^k, X),$$

with Δ_a^k the affine k-simplex, and $NPlot(X)$ is the nerve of the plot category of X.

So $\int X$ is the **smooth homotopy type** of X. This is different than its underlying D-topology.

Now let us use the shape functor to give a proof of PIZ's theorems in the context of ∞-stack cohomology:

Theorem[Min24b]:

$$\check{H}^k_\infty(T_K, \mathbb{R}^\delta) \cong H^k_{\mathrm{grp}}(K, \mathbb{R}),$$

where $H^k_{\mathrm{grp}}(K, \mathbb{R})$ is the group cohomology of K with coefficients in $\mathbb{R}$.

The Shape of a Diffeological Space

Recall that $T_K = \mathbb{R}^n/K$, where $K \subset \mathbb{R}^n$ is a diffeologically discrete subgroup. Then we have a diffeological principal -bundle

$$\begin{array}{ccc} K & \hookrightarrow & \mathbb{R}^n \\ & & \downarrow{\scriptstyle \pi} \\ & & T_K \end{array}$$

This bundle $\pi : \mathbb{R}^n \to T_K$ is classified by the ∞-stack $NDiffPrin_K$ of diffeological principal K-bundles.

We obtain a homotopy pullback diagram

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & * \\ {\scriptstyle \pi}\downarrow & \lrcorner \ \text{ho} & \downarrow \\ T_K & \xrightarrow{g_\pi} & NDiffPrin_K \end{array}$$

The Shape of a Diffeological Space

Now we use a powerful result of Sati-Schreiber.

Proposition[SS22]: If K is a simplicial set and A, B are simplicial presheaves, then we have a weak equivalence

$$\int (A \times^{\mathrm{ho}}_{\mathrm{Disc}(K)} B) \simeq \textstyle\int A \times^{\mathrm{ho}}_{K} \textstyle\int B,$$

of simplicial sets.

Now we have a weak equivalence

$$NDiffPrin_K \simeq \mathbf{B}K = \mathrm{Disc}(N[K \rightrightarrows *]),$$

see [FSS12].

The Shape of a Diffeological Space

So we end up with a homotopy pullback square

$$
\begin{array}{ccc}
* \simeq \int \mathbb{R}^n & \longrightarrow & * \\
{\scriptstyle \int \pi}\downarrow & \lrcorner\ \text{ho} & \downarrow \\
\int T_K & \xrightarrow[\int g_\pi]{} & N[K \rightrightarrows *]
\end{array}
$$

Which means that the map $\int g_\pi$ has contractible homotopy fibers, by the long exact sequence of homotopy groups.

Now we know that T_K is diffeologically connected, so $\int T_K$ is as well, and $N[K \rightrightarrows *]$ is also connected. Thus $\int g_\pi$ is a weak equivalence of simplicial sets. Thus the smooth homotopy type of T_K is $N[K \rightrightarrows *]$.

Now we have

$$\begin{aligned} \check{H}^k_\infty(T_K, \mathbb{R}^\delta) &:= \pi_0 \mathbb{R}\check{\mathbb{H}}(T_K, \mathrm{Disc}(\mathbb{R}^\delta)) \\ &\cong \pi_0 \mathbb{R}\check{\mathbb{H}}(\textstyle\int T_K, \mathbb{R}^\delta) \\ &\cong \pi_0 \mathbb{R}\check{\mathbb{H}}(N[K \rightrightarrows *], \mathbb{R}^\delta) \\ &\cong \pi_0 \underline{\mathrm{sSet}}(\mathrm{B}K, \mathrm{B}^k \mathbb{R}^\delta). \end{aligned}$$

The last line is precisely $H^k_{\mathrm{grp}}(K, \mathbb{R}^\delta)$.

The Higher Obstructions

The Higher Obstructions

Now let us return to the Čech-de Rham Obstruction.

We will approach this in a very different way. Everything revolves around the following ∞-stack

$$\mathrm{B}^k_\nabla \mathbb{R} = \mathrm{DK}[\Omega^0 \xrightarrow{d} \Omega^1 \to \cdots \to \Omega^k_{\mathrm{cl}}]$$

This is called the **Pure Deligne Stack**. Let us get a feel for this ∞-stack when $k = 1$.

The Higher Obstructions

If X is a diffeological space, then let us compute $\check{H}^0_\infty(X, \mathsf{B}^1_\nabla \mathbb{R})$. By running through a lot of machinery, we end up analyzing the following double complex

$$\begin{array}{ccccc}
\Omega^0(QX_0) & \xrightarrow{\delta} & \Omega^0(QX_1) & \xrightarrow{-\delta} & \Omega^0(QX_2) \\
d\downarrow & & d\downarrow & & d\downarrow \\
\Omega^1(QX_0) & \xrightarrow{-\delta} & \Omega^1(QX_1) & \xrightarrow{\delta} & \Omega^1(QX_2)
\end{array}$$

The Higher Obstructions

Which is

$$\begin{array}{ccccc} \prod_{p_0} \Omega^0(U_{p_0}) & \xrightarrow{\delta} & \prod_{f_0:U_{p_1}\to U_{p_0}} \Omega^0(U_{p_1}) & \xrightarrow{-\delta} & \prod_{(f_1,f_0)} \Omega^0(U_{p_2}) \\ d\downarrow & & d\downarrow & & d\downarrow \\ \prod_{p_0} \Omega^1_{\mathrm{cl}}(U_{p_0}) & \xrightarrow{-\delta} & \prod_{f_0:U_{p_1}\to U_{p_0}} \Omega^1_{\mathrm{cl}}(U_{p_1}) & \xrightarrow{\delta} & \prod_{(f_1,f_0)} \Omega^1_{\mathrm{cl}}(U_{p_2}) \end{array}$$

A 0-cocycle is an element (A, g) of

$$\prod_{p_0} \Omega^1_{\mathrm{cl}}(U_{p_0}) \oplus \prod_{f_0} \Omega^0(U_{p_1})$$

such that $-\delta g = 0$ and $dg = -\delta A$.

Unravelling this means that (A, g) is a pair

- $g : QX_1 \to \mathbb{R}$, equivalently, a collection of maps $g_{f_0} : U_{p_1} \to \mathbb{R}$ for every map $f_0 : U_{p_1} \to U_{p_0}$ of plots of X,
- a 1-form A_{p_0} on U_{p_0} for every plot $p_0 : U_{p_0} \to X$,
- such that $\delta g = 0$, equivalently $g_{f_0 f_1} = f_1^* g_{f_0} + g_{f_1}$, for every composable pair of plots maps f_0, f_1,
- such that $dg = -\delta A$, equivalently

$$A_{p_1} = f_0^* A_{p_0} + dg_{f_0}.$$

Notice that this is the usual form of a cocycle description of a connection on a principal $\mathbb{R}$-bundle.

Thus $\mathsf{B}^1_\nabla\mathbb{R}$ is the classifying ∞-stack for principal $\mathbb{R}$-bundles with connection!

In fact, for any Lie group G, there is a classifying stack for principal G-bundles with connection $\mathsf{B}^1_\nabla G$, defined very similarly. Plugging in a diffeological space, we get a **cocycle description for connections on diffeological bundles**! Let $\mathfrak{g}$ denote the Lie algebra of G, then the cocycle equation is:

$$A_{p_1} = \mathrm{Ad}^{-1}_{g_{f_0}}(f_0^* A_{p_0}) + g_{f_0}^*(\mathrm{mc}(G)).$$

Furthermore this cocycle description of connections is equivalent to Waldorf's definition of diffeological connection given in [Wal12].

Theorem[Min24b, Theorem A.3]: If G is a Lie group with Lie algebra $\mathfrak{g}$, and $\pi : P \to X$ is a diffeological principal G-bundle over a diffeological space X, then there is an equivalence

$$\mathbf{Coc}_{\nabla}(X, G) \to \mathbf{Wal}_G(X)$$

between the groupoids of cocycle connections and waldorf connections.

Okay, so $\mathsf{B}^k_\nabla \mathbb{R}$ classifies diffeological principal $\mathbb{R}$-bundles with connection when $k = 1$. For higher k, this ∞-stack classifies what are called **bundle gerbes** with connection.

Rather than define them geometrically and show the equivalence with cocycle descriptions, lets just define them using the cocycle descriptions.

To not go too long on time here, I am just going to show that you can get a huge diagram of relationships between a bunch of different, interesting ∞-stacks.

$$
\begin{array}{ccccccc}
* & \longrightarrow & \mathsf{B}^k\mathbb{R}^\delta & \longrightarrow & * & \longrightarrow & * \\
\downarrow & 1 & \downarrow & 2 & \downarrow & 3 & \downarrow \\
* & \longrightarrow & \mathsf{B}^k_\nabla\mathbb{R} & \longrightarrow & \Omega^{k+1}_{\mathrm{cl}} & \longrightarrow & \Omega^{k+1} \\
 & & \downarrow & 4 & \downarrow & 5 & \downarrow \\
 & & \mathsf{B}^k\mathbb{R} & \longrightarrow & \mathsf{B}^k\Omega^1 & \longrightarrow & \Omega^{1\leqslant\bullet\leqslant k+1} \\
 & & \downarrow & 6 & \downarrow & 7 & \downarrow \\
 & & * & \longrightarrow & \mathsf{B}^{k+1}\mathbb{R}^\delta & \longrightarrow & \mathsf{B}^{k+1}_\nabla\mathbb{R}
\end{array}
$$

Furthermore, we can show that each square here is a homotopy pullback square. This means that for any diffeological space X we get a sequence of fibrations of spaces

$$* \to \mathbb{R}\check{\mathbb{H}}(X, \mathsf{B}^k\mathbb{R}^\delta) \to \mathbb{R}\check{\mathbb{H}}(X, \mathsf{B}^k_\nabla\mathbb{R}) \to \mathbb{R}\check{\mathbb{H}}(X, \Omega^{k+1}_{\mathrm{cl}}) \to \mathbb{R}\check{\mathbb{H}}(X, \mathsf{B}^{k+1}\mathbb{R}^\delta)$$

Taking π_0 of this sequence of fibrations gives us an exact sequence of vector spaces

Theorem[Min24b, Cor. 7.2]: For any diffeological space X and $k \geqslant 1$, there is an exact sequence of vector spaces

$$0 \to \check{H}^k_\infty(X, \mathbb{R}^\delta) \to \check{H}^0_\infty(X, \mathbf{B}^k_\nabla \mathbb{R}) \to \Omega^{k+1}_{\mathrm{cl}}(X) \to \check{H}^{k+1}_\infty(X, \mathbb{R}^\delta).$$

This looks kind of like PIZ's sequence, but isn't quite. The issue is that there is a $\Omega^{k+1}_{\mathrm{cl}}$ instead of H^{k+1}_{dR} and we are missing a term in front.

Note: This sequence was also obtained by David Jaz Myers using entirely different methods in Homotopy Type Theory [Mye24].

Right away we can compute the pure differential cohomology of the irrational torus T_α.

Theorem[Min24b, Th 7.3]

$$\check{H}^0_\infty(T_\alpha, \mathsf{B}^1_\nabla \mathbb{R}) \cong \begin{cases} \mathbb{R}^2, & k = 1, \\ \mathbb{R}, & k = 2, \\ 0, & k > 2. \end{cases}$$

By playing around with these cocycles, we can obtain another series of exact sequences that includes de Rham cohomology, but at the cost of losing injectivity in the beginning:

Theorem [Min24b, Theorem 7.5] Given a diffeological space X and $k \geqslant 1$, there is an exact sequence of vector spaces

$$\check{H}^k_\infty(X, \mathbb{R}^\delta) \to \check{H}^k_{\mathrm{conn}}(X, \mathbb{R}) \to H^{k+1}_{\mathrm{dR}}(X) \to \check{H}^{k+1}_\infty(X, \mathbb{R}^\delta)$$

where $\check{H}^k_{\mathrm{conn}}(X, \mathbb{R}) \subseteq H^{k+1}_{\mathrm{dR}}(X) \oplus \check{H}^k_\infty(X, \mathbb{R})$ consists of the subgroup of those elements (F, g) of bundle gerbes g on X equipped with a connection ω whose curvature form $d\omega$ has cohomology class $[d\omega] = F$.

When $k = 1$, this theorem simplifies beautifully, and we get back all of PIZ's terms. It turns out that if (A, g) describe a principal $\mathbb{R}$-bundle with connection, then $[dA]$ completely determines the whole class (A, g), in the sense that $\check{H}^1_{\mathrm{conn}}(X, \mathbb{R})$ is isomorphic to the group of isomorphism classes of principal $\mathbb{R}$-bundles that **admit** a connection.

Theorem[Min24b, Th 7.7] Given a diffeological space X, there is an exact sequence of vector spaces

$$0 \to H^1_{\mathrm{dR}}(X) \to \check{H}^1_\infty(X, \mathbb{R}^\delta) \to \check{H}^1_{\mathrm{conn}}(X, \mathbb{R}) \to H^2_{\mathrm{dR}}(X) \to \check{H}^2_\infty(X, \mathbb{R}^\delta)$$

The Higher Obstructions

Summary: Using the machinery of higher topos theory, we obtained a homotopical framework to manipulate diffeological spaces and define ∞-stack cohomology for them.

This provided us with a handy toolbox to prove that

$$\check{H}^k_\infty(T_K, \mathbb{R}^\delta) \cong H^k_{\mathrm{grp}}(K, \mathbb{R}^\delta),$$

and to obtain an analogue of PIZ's theorem to all dimensions

$$\check{H}^k_\infty(X, \mathbb{R}^\delta) \to \check{H}^k_{\mathrm{conn}}(X, \mathbb{R}) \to H^{k+1}_{\mathrm{dR}}(X) \to \check{H}^{k+1}_\infty(X, \mathbb{R}^\delta).$$

Thank you so much for your patience.

Questions?

Comments?

Feel free to email me at eminichiello67@gmail.com

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